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FIXED-INCOME PORTFOLIO OPTIMIZATION

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FIXED-INCOME PORTFOLIO OPTIMIZATION

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REPORT

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To my mother and father, for their enduring support and affection.

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This Report marks the end of one chapter in my life and the beginning of another; it presents me with an opportunity to acknowledge those who have helped me along the way.

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ABSTRACT

FIXED-INCOME PORTFOLIO OPTIMIZATION

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The fixed maturity, pricing and cash flow characteristics of fixed-income instruments like bonds distinguish them from equities and complicate the application of mean-variance optimization techniques to bond portfolio management. This report examines the challenges involved and reviews some of the theoretical term structure models and empirical estimation methods that have been proposed to address them. An empirical study is conducted which finds evidence of increased interest rate volatility, which affirms the need for a portfolio approach in fixed-income investing. An optimal portfolio of bond funds constructed using the Markowitz method is found to provide the best risk-return profile over the chosen study period, suggesting the viability of this approach as an alternative to holding bonds.

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I. INTRODUCTION

Modern portfolio theory (MPT) is widely acknowledged as being one of the most significant developments within finance in the past half century. It has found widespread application – explicitly, via its use in developing quantitative asset allocation models and implicitly, as a framework to guide asset managers' thinking on portfolio construction. However, this has been largely restricted to the equities space; fixed-income practitioners have had limited use for these methods.

This is primarily due to the fundamental differences between these two asset classes – especially those pertaining to their maturities and price paths over time – which preclude the direct application of the mean-variance model to bonds. The advent of sophisticated term structure models has helped, especially in the pricing and risk-management of fixed-income derivatives. However, most fixed-income managers continue to use relatively crude techniques like indexing and managing against benchmarks to structure their portfolios.

This report attempts to explain why this is the case. Beginning with a review of basic bond terminology, it examines the shortcomings of MPT when applied to the fixed-income markets. A major problem is that of estimating the input parameters of the Markowitz model – the time dependence of bond returns means that we cannot use simple historical estimation based on the assumption of stationarity. Conditional prediction models called *term structure models* are particularly useful in this context; a fairly detailed treatment of modern term structure models is thus presented. Most of these models require initial estimates of the term structure to calibrate their parameters. A review of the statistical techniques used to generate these estimates is also provided. The attractiveness of fixed-income instruments as investments extends beyond the expected returns – their deterministic payment streams also make them suitable for other objectives like servicing liabilities. The section on professional strategies describes some such objectives and the standard industry practice in meeting them.

Finally, an empirical study is conducted to assess the relative merits of some of the methods reviewed. Evidence is found of heightened interest rate volatility over the past two decades, which points to the potential benefits of diversification. While bond indexing is found to provide a viable alternative to holding individual bonds, an optimal portfolio of bond funds constructed using the Markowitz method is found to provide the most attractive risk-return profile. This suggests that portfolio optimization might have a role to play in fixed-income portfolio management – it can be used to structure portfolios of bond funds and conceivably even those of bonds as well. A modified version of the method based on term structure models would be required for the latter; although methods of this type have been proposed, they are sparingly used, perhaps because they have (so far) failed to justify the additional complexity and expense involved. Bridging this practicality gap remains one of the main challenges facing fixed-income researchers today.

II. LITERATURE SURVEY

The long history of fixed-income instruments and their continued relevance over time has meant that this is a very well-studied area within finance. This section provides a brief tour of the academic/professional literature as it relates to the subject matter of this report.

Perhaps the earliest attempt to adapt the mean-variance model for use in bond portfolio optimization was made by Cheng (1962)[8], who analyzed the effect of reinvestment risk on bond portfolios. He modeled the tradeoff between rolling-over short term investments and investing at the spot rate until maturity (or the end of the investment horizon), using probability beliefs on future reinvestment rates (and thus the term structure) as inputs; these beliefs were based on empirical data on historical interest rate movements. Bradley/Crane (1972)[5] improved on this by providing a dynamic bond portfolio selection formulation. Both these methods, while worthy first steps, were relatively simplistic and suffered from extensive data requirements, which limited their practical utility.

The first major breakthrough in this area was the Vasicek model (1977)[37] – it heralded the birth of dynamic term structure modeling, which involves representing the evolution of interest rates by stochastic processes. Two popular extensions – the Cox/Ingersoll/Ross model (1985)[10] and the Hull/White model (1994)[18] were to gain considerable acclaim and are discussed in this report. The later Heath/Jarrow/Morton model (1992)[15], which modeled the entire yield curve, also finds mention here. Other notable works include those of Dothan (1978)[11], Cox/Huang (1989)[9] and Black/Karasinski (1991)[2].

Unfortunately, the availability of viable term structure models does not automatically solve the bond portfolio selection problem. Indeed, a parallel branch of research is dedicated to examining the applicability of term structure models to bond portfolio selection. Notable works in this area include Wilhelm (1992)[39] and Korn/Koziol (2006)[22].

Another area of study – Term Structure Estimation – uses empirical data and regression analysis to estimate the yield curve based on a sample of observed bond prices. Three techniques of this type – McCulloch (1971)[27], Fama/Bliss (1987)[14] and Extended Nelson-Siegel (1996)[3] are presented here.

Finally, the following books proved particularly useful as general references throughout the course of this study – *Investments* by Bodie/Kane/Marcus (2008)[4] for basic bond concepts, *Bond Portfolio Optimization* by Puhle (2008)[33] for theoretical term structure models and *Advanced Fixed-Income Portfolio Management* by Fabozzi/Fong (1994)[13] for professional perspectives and empirical analysis. Each of them is referenced several times in this report.

III. BOND TERMINOLOGY REVIEW

- **Bonds** are fixed-income instruments which represent the holder's claims on a fixed series of future payments. The buyer of a bond (the lender) lends the issuer (the borrower) a certain amount of money; in return, the issuer commits to repaying the principal amount at a certain date in the future, plus periodic interest payments in the interim.
- A bond's parameters include its **Par Value** (the amount on which the issuer pays interest; this must be repaid at the end), **Maturity Date** (the date on which the principal is repaid), **Coupon Rate** (the interest rate that the issuer pays) and **Coupon Payment Frequency** (the periodic time intervals at which interest is paid in the form of coupon payments). All of these parameters are known prior to issue.
- **Bond Price:** Since the bond's future cash flows are fixed, the price of the bond represents what this stream of payments is worth to the holder today. Time value of money requires that we compute the present value of these payments by discounting at a certain rate of interest. This gives the following expression,

$$Bond\ Value = \sum_{t=1}^{T} \frac{CF_t}{(1+r)^t} = \sum_{t=1}^{T} \frac{Coupon_t}{(1+r)^t} + \frac{Par\ Value}{(1+r)^T}$$

Where, CF_t = cash flow at time t, T = maturity date, r = interest rate

- The only exogenous variable in the above equation is the interest rate. In fact, interest rates are the primary determinants of bond prices over time. If, for convenience, we assume a constant interest rate over time (as above), it should be clear that the bond value is a decreasing function of *r*, as shown in Figure 1.
- Regardless of whether we use one or more interest rates, the selection of the appropriate rate(s) is a subjective process. Market rates on instruments having similar characteristics and credit quality are typically used. The rates on

Treasury bonds (some of which are zero-coupon bonds (ZCBs) selling at a discount from par) are widely used as benchmarks.

- In reality, bond prices rarely match the values expected from using the above equation. Considerations of default, liquidity, taxation, supply-demand, call provisions, embedded options, contingent claims, etc. combine to determine the price the bond ultimately trades at.
- **Yield-to-Maturity (YTM):** It is the return obtainable on a bond by buying it now and holding to maturity (assuming no default).
    - YTM is a theoretical construct - notice that when we buy a bond, we are not quoted a rate of return (except the coupon rate, which does not account for the total return obtainable if the bond trades at any value other than par). The rate of return or **Yield** is implicit and is a function of the coupon rate, the purchase price of the bond and the time remaining to maturity.
    - Bonds usually trade at prices that make the YTM equal to the relevant market interest rates. This is due to the No Arbitrage characteristic of an efficient market: if superior returns could be achieved at no additional risk by holding one instrument vs. another, the price of the former would be bid up to a point where their returns would be the same.
    - Therefore, YTM can be viewed as a proxy for market interest rates. If YTM > coupon rate, additional compensation would be required to invest in the bond – this is achieved by pricing it at a *discount* to par, the additional compensation coming in the form of capital appreciation (from purchase price to par value) at maturity. The reverse is also true: when coupon rate > YTM, the bond trades at a *premium*. The variation in prices of discount and premium bonds with time (remaining to maturity) are shown in Figure 2.

Figure 1 **Price-yield relationship for bonds**

(Price curve for a 8%, semiannual coupon, 30-year maturity bond, par $1000)

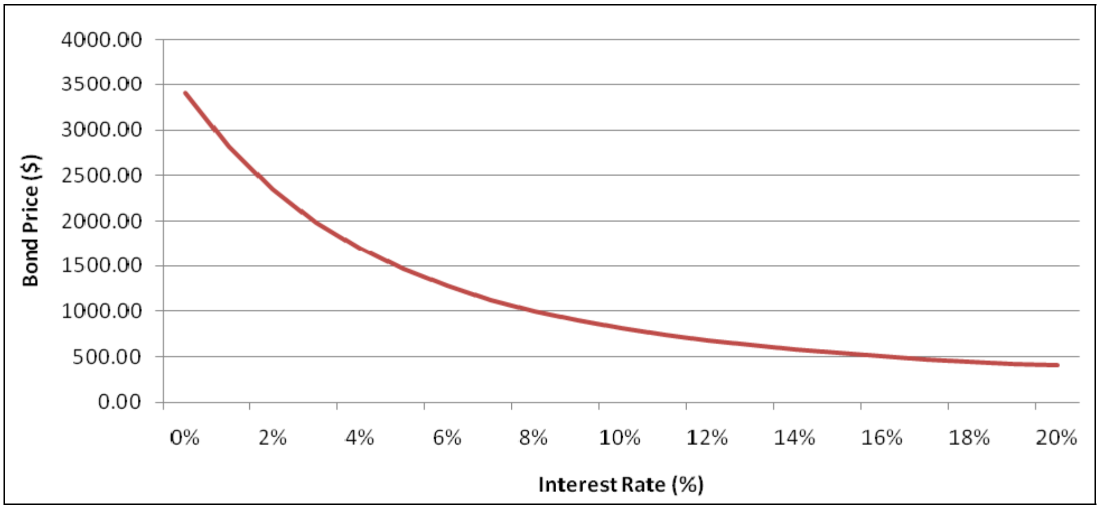
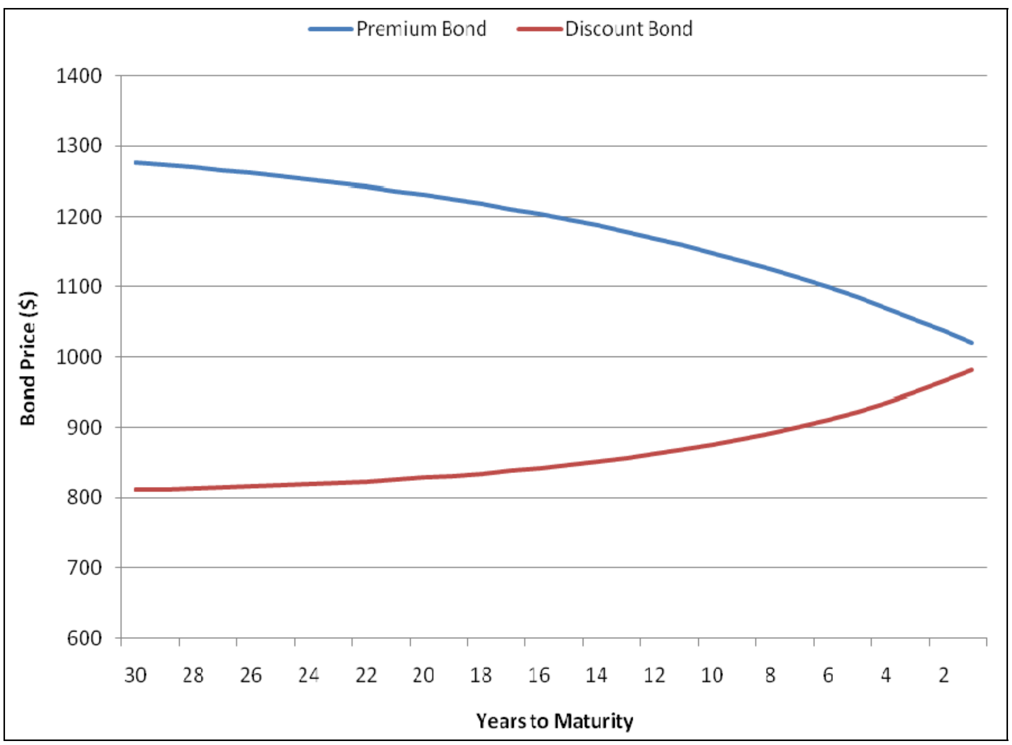


Figure 2 **Variation of bond prices over time**

(Price curves for a 8%, semiannual coupon, 30-year maturity bond, par $1000 at YTM 6% and 10%)



- YTM may also be thought of as being the internal rate of return of a bond, i.e., the interest rate which, when used to discount the future payments, results in a net present value of zero.
- Conventional YTM has two major flaws: (1) It assumes reinvestment of the coupon payments at the same rate of interest – if this does not hold, the realized return will differ from the expected return; (2) The likelihood of default often causes the market to price bonds based on the expectation of recouping less than the par value – in such a scenario, the *stated YTM* would exceed the *expected YTM*.

- In reality, interest rates do not remain constant over time. Bonds having different maturities sell at different yields and their cash flows must be discounted by different interest rates, based on the times at which they occur. The **Spot Rate** is the interest rate that is quoted for immediate settlement and delivery. For bonds, this means the rate that prevails today for a bond of a certain maturity. The **Short Rate** for a given time interval (say 6 months) refers to the interest rate for that time interval that is available at different points in time. Since future interest rates are uncertain, future short rates are speculative and need not be borne out in reality; they are thus called **Forward Rates**.
- The relationship between these interest rates is given by the Law of One Price, which states that all else being equal, two securities having the same cash flows must be priced the same. This arises from the need for no arbitrage described earlier and gives rise to the following relationship,

$$(1+f_n)=\frac{(1+y_n)^n}{(1+y_{n-1})^{n-1}}$$

Where, y_n = n-period spot rate, f_n = forward rate in period n

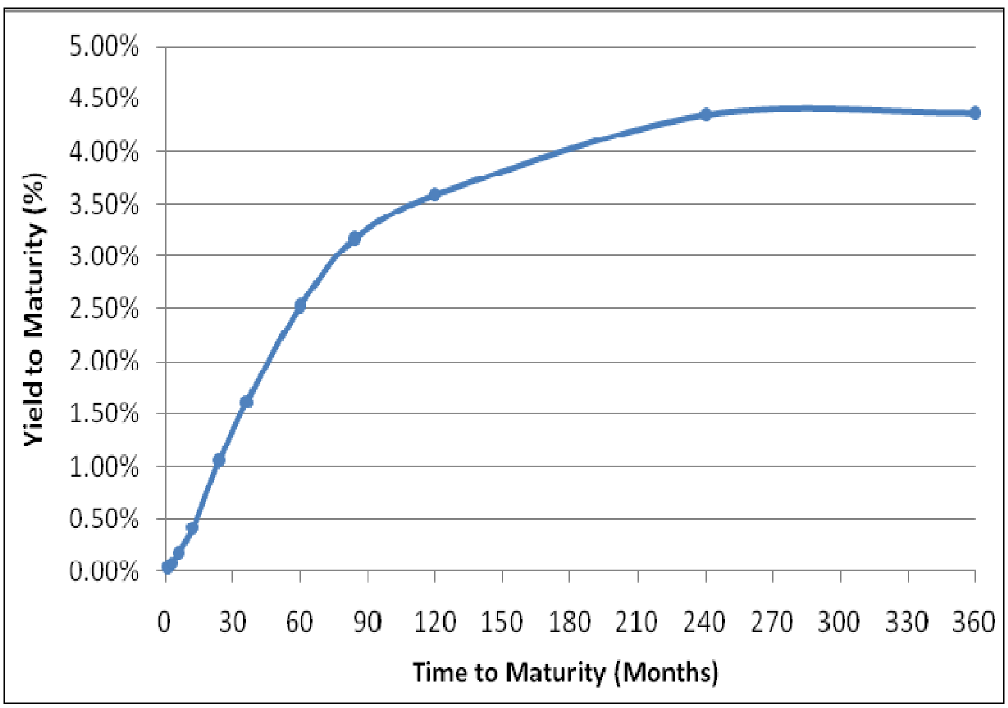
f_n is sometimes denoted ${}_{n-1}f_n$ for more clarity, with *n-1*, *n* denoting the period start and end times.

The n-period spot rate y_n is thus the geometric mean of (1 plus) the $(n-1)$-period spot rate y_{n-1} and the n-period forward rate f_n, or equivalently, of the 1-period spot rate (y_1) and the forward rates from periods $2 \rightarrow n$ ($f_2 \ldots f_n$).

- The bond **Yield Curve** (also known as the **Term Structure of Interest Rates**) is a plot of YTMs vs. maturities for bonds in the same asset class and having the same credit quality. Since each bond has a different yield, the choice of a representative yield curve is subjective. Perhaps the most widely observed yield curve is that of U.S. Treasury securities – it is typically constructed using on-the-run (the latest issued and thus most liquid securities in each maturity class) Treasuries of 3 month, 2 yr, 5 yr, 10 yr and 30 yr maturities. The current Treasury yield curve is shown in Figure 3.

Figure 3 **U.S. Treasury Yield Curve – Oct 26, 2009**

(YTMs for 1 mo, 3 mo, 6 mo, 1 yr, 2 yr, 3 yr, 5 yr, 7 yr, 10 yr, 20 yr, 30 yr Treasuries)



- Since the spot rate y_n is a geometric mean of y_{n-1} and f_n, it should be intuitive that $y_n > y_{n-1}$ implies $f_n > y_{n-1}$. This results in an upward-sloping yield curve (as is the case in Figure 3). The opposite is also true: if $f_n < y_{n-1}$, then the yield curve is downward sloping. Since f_n is speculative, it follows that the term structure reflects market expectations of (unknown) future interest rates.
- There are three (main) theories of the term structure – the **Expectations Hypothesis**, the **Liquidity Preference Hypothesis** and the **Market Segmentation Hypothesis**. Without going into the details, it is sufficient to note that given the variety of factors affecting interest rates, backing out of the yield curve to try and unearth the underlying drivers of interest rates is a difficult task.
- **Duration**: It is necessary to quantify the sensitivity of bonds to interest rate changes. The most common metric used for this is the bond's *Macaulay Duration* – it is the weighted average of the times to each cash flow (coupon or principal payment), the weights being the proportion of the bond's overall value that resides in that payment. Mathematically,

$$\textit{Macaulay Duration},\ D = \sum_{t=1}^{T} \left(w_t \times t \right)$$

$$w_t = \left(\frac{\frac{CF_t}{(1+y)^t}}{\textit{Bond Value}} \right)$$

Where, w_t = weight of CF at time t, T = periods to maturity, y = yield to maturity

The bond's price sensitivity to interest rate changes is then given by,

$$\frac{\Delta P}{P} = -D \times \left[\frac{\Delta(1+y)}{(1+y)} \right]$$

Where, ΔP/P = Percentage change in Price, D = Duration

This equation is often modified to obtain the *Modified Duration*,

$$\textit{Modified Duration}, D^* = \frac{D}{(1+y)} \quad \Rightarrow \quad \frac{\Delta P}{P} = -D^* \times \Delta y$$

Duration is a key concept in fixed-income portfolio management for 3 reasons: (1) It is a simple summary statistic of the effective average maturity of the portfolio; (2) It is a measure of the interest rate sensitivity of a portfolio; (3) It is an essential tool for immunizing portfolios from interest rate risk.

- **<u>Convexity</u>**: While duration is a useful concept, it is merely an approximation; its accuracy decreases as the magnitude of the yield change increases. This can be explained by the fact that, in reality, the relationship between percentage change in price (*ΔP/P*) and percentage change in yield (*Δy/y*) is convex, whereas duration predicts a linear (inverse) relationship between the two. The curvature of the price yield curve is characteristic of a particular bond and is known as its convexity. It is therefore necessary to adjust the duration expression for convexity, in order to be able to accurately predict *ΔP/P* over all ranges of *Δy/y*.

$$\frac{\Delta P}{P} = -(D^* \times \Delta y) + \left[\frac{1}{2} \times Convexity \times (\Delta y)^2\right]$$

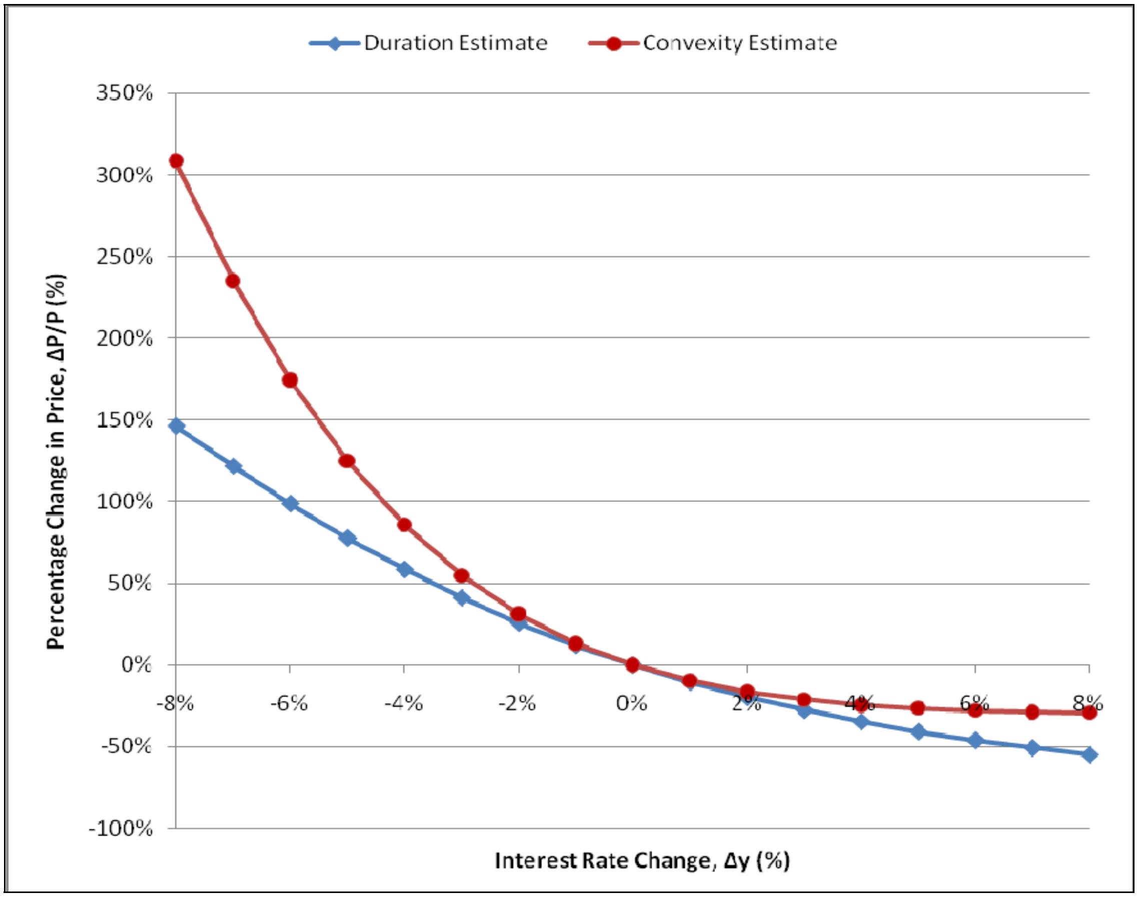
Where,

$$Convexity = \frac{1}{P(1+y)^2} \times \sum_{t=1}^{T} \left[\frac{CF_t}{(1+y)^t} \times (t^2 + t)\right]$$

The duration and convexity of a coupon bond are plotted in <u>Figure 4</u>. Notice that duration always underestimates the price of the bond – it underestimates the percentage price rise when interest rates fall and overestimates the percentage price drop when interest rates rise.

Figure 4 **Price Change Estimates - Duration vs. Convexity**

(Estimates for a 8% annual coupon, 30-year maturity bond, par $1000 at YTM 8%)



IV. CHALLENGES IN BOND PORTFOLIO OPTIMIZATION

We now examine the application of Modern Portfolio Theory (MPT) to the fixed-income markets. This theory is based in large part on the mean-variance optimization framework proposed by Markowitz (1952)[25], so we begin with a brief review of this model, followed by an assessment of its applicability to bonds.

4.1 The Markowitz Mean-Variance Model

This framework presents a mathematical formulation of the concept of diversification in investing. The model is static in that it is designed to construct a portfolio today, which is then held till the end of the investment horizon; no rebalancing is assumed in the interim. The model assumes that investors are risk-averse, meaning that if there are two assets with the same expected returns investors will prefer the less risky one. The implication is that higher expected returns may be achieved only by assuming greater risk. Further, it assumes that investors care only about the expected (terminal) value and variance of portfolio returns, which are defined as follows:

$$E(R_P) = \sum_{i=1}^{N} w_i \times E(R_i)$$

$$\sigma_P^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{N} \sum_{j=i+1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij}$$

Where, $E(R_p)$ *= Portfolio expected return* $\quad$ $\sigma_p^2 = Var(R_p)$ *= Portfolio variance of returns*

N *= number of assets in portfolio* $\quad$ w_i *= weight of asset i, (I = 1…N)*

σ_i *= Standard deviation of asset i* $\quad$ ρ_{ij} *= correlation between returns of assets (i,j)*

Historical returns provide expected values of future returns and the variance of past returns is used as a proxy for risk. With these assumptions, the problem is formulated as:

$$\min_{N}(\sigma_P^2)$$

$$\textit{Subject to:} \quad \sum_{i=1}^{N} w_i \times E(R_i) = \mu = E(R_p)$$

$$\sum_{i=1}^{N} w_i = 1$$

Where, μ = Target rate of (expected) return

The objective is thus to minimize the variance for a given expected rate of return. Additional constraints such as short-sale restrictions, position limits, etc. may be imposed, if needed.

Realistically, the portfolio selection process should span multiple time-periods; however, existing multi-period modeling frameworks tend to be more complex and less tractable than the single period formulation. Mossin (1968)[31] suggests that a multi-period problem be treated as a series of single-period problems if (and only if) the following conditions are fulfilled: (1) Returns are i.i.d; (2) The investor's utility function has a constant relative risk aversion (i.e., relative risk aversion is independent of wealth). The latter implies that the investor cares only about $E(R_p)$ and σ_p^2, which is at odds with the expected utility maximization objective suggested by conventional economic theory. In order to reconcile these objectives, it is necessary to assume that investors have a quadratic utility function.

The major insight provided by this model was that investors may reduce portfolio risk simply by holding assets that are not perfectly positively correlated. The optimal portfolio will be the tangency portfolio on the Efficient Frontier.

4.2 Applying the Markowitz Method to Bonds

While mean-variance optimization has become a cornerstone of equity portfolio management, much less is known about portfolio optimization in the bond markets. Korn/Koziol ascribe this to the relative historical stability of interest rates, which they believe might have rendered an explicit portfolio approach unnecessary in the past. Motivations notwithstanding, applying the Markowitz method to bonds remains a challenging problem due to the systematic differences between stocks and bonds. In particular, the fixed maturity, periodic coupon payments, default risk and interest-rate dependence of bonds distinguish them from stocks and complicate the estimation of the input parameters of the mean-variance model.

The fixed maturity of bonds means that all bonds having maturities less than the investment horizon (time T) will not exist at T. Even if we restricted our holdings solely to ZCBs (thereby eliminating the need to reinvest periodic coupon payments), this would require an assumption about reinvestment of cash flows received before T. This assumption must provide an optimal allocation without requiring rebalancing at each coupon payment; otherwise, the model would become a dynamic programming problem in which a decision taken at any point in time (t) must anticipate future decisions. The easiest way to overcome this is to assume the future interest rates; one such assumption is made by Wilhelm (1992), who assumes that all cash flows received at times $t < T$ are reinvested at the current spot rate $R(t,T)$ until T. While such assumptions do simplify the problem, they seldom lead to optimal solutions.

Another problem is that of determining the expected bond prices at the investment horizon, $E[P(t,T)]$. Bond prices are functions of time and interest rates; they become non-random at maturity (bond price at $t = T$ is simply the par value). These properties prevent the use of the historical time series of prices or conventional modeling techniques like geometric Brownian motion, both of which are used for stocks.

Two additional challenges are identified by Fabozzi/Fong, who consider the major problem to be that of constructing the covariance matrix for bonds. The finite life of a bond means that its covariance with other bonds changes over time, if for no other reason than the shortening of the time remaining to maturity. Further, rate changes can be accompanied by changes in the shape (slope) and direction (upward/downward) of the yield curve; direction changes can alter, and in some cases, reverse the covariance relationships between securities.

Korn/Koziol point to the variation in moments over time as being problematic: "In a high interest-rate period the returns of bonds are supposed to be higher than in a low interest-rate period. Therefore, if interest rates decline for a period, the average return during this period seems to overstate the return during the succeeding period with (probably) lower interest rates." This is debatable, since it could be argued that this holds for stocks as well – returns in bull markets overstate those in bear markets and vice versa. A potential difference is that interest rate regimes are longer and their effects more pervasive than stock market cycles – they impact all classes of bonds, whereas equity market cycles tend to affect some sectors more than others.

Finally, Meindl/Primbs (2006)[29] describe the layer of complexity added by the possibility of bond defaults – they observe that the future prices of corporate bonds tend to have bimodal distributions with one mode corresponding to the default state and the other to the no-default state. This makes bond portfolio optimization models much less tractable than their equity counterparts.

From the above discussion, it should be clear that the unique characteristics of bonds mean that traditional portfolio optimization models like the Markowitz method cannot be used directly in constructing portfolios of these securities; several modifications are required, which are detailed in future sections.

V. TERM STRUCTURE MODELING

Since the historical time series of bond prices does not provide consistent bond portfolio selection parameters, there is a need for a theoretical model to explain the evolution of bond prices over time.

The Duration model put forth by Macaulay (1938)[24] was one of the first to capture (some of) the dynamics of the term structure. Duration proved to be a useful measure of interest rate sensitivity and was readily adopted by practitioners. However, it did suffer from one serious shortcoming – by using a single value of y (YTM) to discount all the cash flows, it assumed a flat term structure with only parallel shifts over time. As Cairns (2004)[6] points out, this is empirically improbable and even if it were true, would present an arbitrage opportunity.

The study of the evolution of the term structure over time with no arbitrage – termed *Dynamic Term Structure Modeling* – has been one of the most heavily researched areas in financial economics over the past three decades. Several viable term structure models have been developed, the most influential of which are discussed here.

5.1 The Dynamics of the Short Rate

An understanding of the mathematical foundation of term structure models is a useful precursor to studying the specific models themselves. A major point of discussion is the modeling of the dynamics of the short rate, which is a random variable (RV) since its value over time changes in an uncertain manner. Most models take the short rate to be a continuous RV whose change over time can be modeled using (continuous-time) stochastic processes, which in turn assumes constant trading. The first step in formulating the model is the specification of the specific stochastic process used. The *standard Wiener process* is a popular choice; it is expressed by the stochastic differential equation (SDE):

$$dr(t) = \mu dt + \sigma dx$$

Where, *dr = change in the short rate,* *dt = length of time interval*
μ = expected direction of rate change *dx = random process*
σ = std. deviation of changes in the short rate

This equation states that the change [*dr(t)*] in the short rate over a very small time interval *dt* depends on the expected direction of the change μ and a random process dx. μ is called the *drift rate;* the randomness in the change of the short rate is due to the RV x in the equation above. In a standard Wiener process, over a very small time interval, $x \sim N(0,1)$, implying that *dr(t)* is directly proportional to *dx*, the proportionality depending on σ. It is also assumed that *dr(t)* over any two different short time intervals are i.i.d. According to Fabozzi/Fong, these assumptions give rise to two properties of the process,

- The expected value of the change in the short rate is equal to the drift (μ). If $\mu = 0$, the expected value of the short rate is its current value.
- The variance of the change in the short rate over some interval T is equal to T and its standard deviation is $\sqrt{T}$.

A special case of the standard Wiener process assumes that both μ and σ are functions of the level of r. This specification, called an *Ito Process*, leads to the following expression:

$$dr(t) = \mu(r,t)dt + \sigma_r(r,t)dz$$

In an Ito process of this form, it is further assumed that the drift rate follows a *mean-reverting process* represented by,

$$\mu(r,t) = \kappa[\theta - r(t)]$$

Where, *κ = speed of mean reversion (κ > 0),*

θ = mean reversion level (long-run stable mean of short rate)

By extension, the short rate also follows a mean reverting process called the *Ornstein-Uhlenbeck (O-U) process* (also known as an *elastic random walk*):

$$dr(t) = \kappa(\theta - r(t))dt + \sigma_r dz(t)$$

Where, $\kappa > 0$; θ, σ_r *=constant*

The drift is positive when $r(t) < \theta$ and negative when $r(t) > \theta$. The process is therefore pulled toward θ, with the magnitude of the pull depending on the mean reversion speed κ. The mean-reversion property is especially appropriate for fixed-income securities; without it, interest rates could permanently drift upwards like stock prices tend to do. The O-U process is Markov (memoryless), i.e., the conditional distribution of its future values is independent of its past values and depends solely on its present state. More specifically, the O-U process is a stationary Markov process with normally distributed increments.

5.2 Interest Rate Modeling Approaches

Puhle (2008) identifies two broad approaches to term structure modeling – models of the short rate of interest and those of the whole yield curve.

5.2.1 Short Rate Models

The most widely used approach is to model the evolution of one instantaneous forward rate, namely the short rate. Since changes in bond prices are highly correlated, changes the short rate are used as a proxy for changes in the whole term structure. Empirical and qualitative justification for this approach has been provided by Martellini/Priaulet/Priaulet (2003)[26] and Litterman/ Scheinkman (2001)[23].

Short rate models are typically characterized by the following: incomplete markets (in which contingent claims cannot be replaced with existing securities), no arbitrage, the money-market account being the only exogenous asset (it yields the current short rate), ZCBs considered to be derivatives on the short rate. As discussed above, a diffusion process (usually the O-U process) is proposed for the evolution of the short rate. Since μ and σ_r only depend on the level of r, the model is always Markov. These models are automatically consistent with the value of the short rates, but not necessarily with other interest rates.

5.2.2 *Whole Yield Curve Models*

These models attempt to model the whole term structure of instantaneous forward rates. ZCBs of all maturities (including money-market accounts, which have instantaneous maturities) are assumed to trade in complete markets. The instantaneous forward rates, denoted $f(t,T)$, are assumed to follow stochastic processes. For a fixed maturity T, this is represented as,

$$df(t,T) = \mu(t,T)dt + s(t,T)dz$$

Where, $\mu(t,T)$ = drift of bond prices, $s(t,T)$ = instantaneous std. deviation,

dz = Brownian motion

This approach is used by Heath/Jarrow/Morton (1992), who show that the drift must follow from the specification of the volatilities and market prices of interest rate risk. Specific term structure models are presented in Section 5.3. Another model of this type was proposed by Hull/White, who represented ZCB prices as,

$$\frac{dP(t,T)}{P(t,T)} = \mu(t,T)dt + \sigma(t,T)dz$$

Where, $P(t,T)$ = price of a zero-coupon bond with maturity T at time t,

$\sigma(t,T)$ = bond price volatility

Notice that the RHS of both these models are essentially identical, implying that any model of zero-coupon bond prices can be converted into an equivalent model of forward rates and vice versa. It has been shown that in a no-arbitrage setting, the drift of bond prices is a function of the short rate r, the volatilities σ and the market prices of interest-rate risk. Volatility is itself an increasing function of time to maturity, so as $t \rightarrow T$ *(i.e.,* $T - t \rightarrow 0$), $\sigma(t, T) \rightarrow \sigma(T, T) = 0$.

Each of these approaches has its merits. Whole yield curve models offer two benefits over short rate models:

- Short rate models depend on the prior specification of μ and σ_r, both of which (μ in particular) can be difficult to estimate. In whole yield curve models, μ follows from arbitrage considerations, meaning that only the volatilities and market price of risk are required to be estimated.
- Since whole yield curve models determine the initial values of the variables being modeled, they are automatically consistent with the initial term structure.

The main disadvantage of whole yield curve models is that they are usually non-Markov and thus computationally intensive.

5.3 Term Structure Models

This section describes four of the most famous theoretical term structure models. The Vasicek and Cox/Ingersoll/Ross models take the evolution of the short rate and risk preferences as being specified. The Hull/White model extends these models to allow them to be calibrated to an observed initial term structure. The Heath/Jarrow/Morton model is a departure from the others in that it takes the initial term structure as given and models the evolution of the whole yield curve.

Given the complexity of each of these models, the emphasis here is on highlighting their salient features. Further details, if required, may be found in the references.

5.3.1 *Vasicek Model*

Vasicek (1977) proposed what is known to be the first arbitrage-free dynamic term structure model. It is a one-factor model in that it assumes that the term structure can be completely determined by a sole random variable – the short rate of interest.

The model is based on three primary assumptions: (1) The spot rate follows a continuous Markov process; (2) The price of a discount bond depends only on the spot rate over its term; (3) The market is efficient, i.e., there are no transaction costs or information asymmetry and all investors act rationally (prefer more wealth to less and make decisions based on all available information). The third assumption implies that investors have homogeneous expectations and that no profitable, risk-less arbitrage is possible.

Vasicek provides both a general interest rate model and a special case; the latter assumes an Ornstein-Uhlenbeck process to represent the evolution of the spot rate *r(t)*, as well as a constant market price of risk given by the following,

$$\lambda(t) = \lambda \qquad \forall t, t = (1, \ldots, T)$$

Where, λ = *market price of risk*

Under these assumptions, the short rate is represented by the following equation,

$$r(T) = r(t)e^{-\kappa(T-t)} + \theta\left(1 - e^{-\kappa(T-t)}\right) + \sigma_r \int_t^T e^{-\kappa(T-u)} dz(u)$$

This leads to the following analytic solution for ZCB prices:

$$P(t,T) = A(t,T) \times e^{-B(t,T)r(t)}$$

Where, $$B(t,T)=\frac{1}{\kappa}\left(1-e^{-\kappa(T-t)}\right)$$

$$A(t,T)=R(\infty)\left(\frac{1}{\kappa}\left(1-e^{-\kappa(T-t)}\right)-(T-t)\right)-\frac{\sigma_r^2}{4\kappa^3}\left(1-e^{-\kappa(T-t)}\right)^2$$

$$R(\infty)=\left(\theta+\lambda\frac{\sigma_r}{\kappa}-\frac{1}{2}\frac{\sigma_r^2}{\kappa^2}\right)$$

R(∞) = Interest rate with infinite maturity

The Vasicek model is analytically tractable, but has several shortcomings as originally proposed. For one, it is Gaussian, so positive probabilities are assigned to negative values of the future short rate. This is an undesirable property, since negative interest rates are impossible in an arbitrage-free market. However, there is no arbitrage since holding cash is not permitted. Further, as Chan (1992)[7] showed, for reasonable values of the parameters, the probability of negative short rates is generally small.

Further, as Bazayit (2004)[1] points out, it implies that the conditional volatility of changes in the interest rate be constant, independent of the level of *r*. In reality, interest rate volatility varies over time, often significantly. In general, it tends to be higher during periods of high interest rates and lower during low interest rate periods. While the level of *r* is not a sufficient estimator of volatility, the absence of any relationship between the two in the Vasicek model is highly unrealistic.

Perhaps the most serious drawback with the model in a portfolio selection context is the perfect correlation of the spot rates, i.e.,

$$Corr\left(R(t,T),R(t,\tau)\right)=1$$

T, τ = Different bond maturities

In reality, ZCB prices are non-linear functions of the short rate, so bonds of different maturities are perfectly, but non-linearly correlated (i.e., $\rho \neq 1$). Finally, the Vasicek model also struggles to perfectly fit the initial term structure of interest rates and to explain the humped volatility structures observed in the marketplace.

Despite these limitations, the Vasicek model was a seminal work in this area, giving rise to numerous extensions and offshoots which succeeded in overcoming some of its failings. The most famous of these – the Cox/Ingersoll/Ross (1985) model and the Hull/White model (1990) are briefly discussed below.

5.3.2 Cox-Ingersoll-Ross (CIR) Model

This model, proposed by John Cox, Jonathan Ingersoll and Stephen Ross in 1985, is an extension of the Vasicek model. Like the original, it is a one-factor model which represents the short-rate with an Ornstein-Uhlenbeck process, with one notable difference: the volatility of the short rate is now expressed as $\sigma\sqrt{r}$, in place of the σ used by Vasicek. This process is referred to as a *mean-reverting square-root diffusion process.* The Vasicek SDE then becomes,

$$dr(t) = \kappa(\theta - r(t))dt + \left(\sigma_r \sqrt{r}\right) dz(t)$$

The main improvement – a result of the altered specification of volatility – is that the possibility of positive probabilities for negative interest rates is avoided altogether if ($2\kappa\theta > \sigma^2$) and minimal otherwise. Other advantages are described by Fabozzi (2001)[12]: (1) It is rigorously consistent with the valuation of fixed-income securities, since it produces both prices and returns; (2) It is defined continuously in maturity, which allows exposures for ZCBs of any maturity to be calculated without need for any interpolation or approximation; (3) The moments (the mean and variance of the single factor return) can be estimated directly by observing the time-series of changes in the short-interest rate; (4) The variance of the process increases with the short rate, which is more realistic. A disadvantage of the CIR model is that it only allows one type of yield curve shift, which means that it is quite limited in the variety of actual yield-curve behavior it can describe.

5.3.3 Hull-White Model

The Hull/White model (1990) is a *two-factor* extension of the Vasicek model, the factors being the short rate (r) and the mean-reversion level (ε). The following relationships are assumed,

$$dr(t) = [\theta + \varepsilon(t) - \kappa_r r(t)]dt + \sigma_r dz_1$$

$$d\varepsilon(t) = -\kappa_\varepsilon \varepsilon(t)dt + (\sigma_\varepsilon \rho)dz_1 + \left(\sigma_\varepsilon \sqrt{1-\rho^2}\right)dz_2$$

Where, θ, σ_r, σ_ε, $\kappa_r > 0$, $\kappa_\varepsilon > 0$ *are constants*

ρ = *correlation coefficient between* r *and* ε; dz_1, dz_2 = *Brownian motions (uncorrelated)*

The market prices of interest rate risk are again assumed constant,

$$\lambda_1(t) = \lambda_2(t) = \lambda$$

This gives the following expression for ZCB prices,

$$P(t,T) = A(t,T) \times e^{-[B_1(t,T)r(t)] - [B_2(t,T)\varepsilon(t)]}$$

Where,

$$A(t,T) = -\theta\left(\frac{-1 + e^{-\kappa_r(T-t)} + (T-t)\kappa_r}{\kappa_r^2}\right) - \frac{1}{2}(T-t)(\lambda_1^2 + \lambda_2^2) + \frac{1}{2}\int_t^T \left(\eta(s)^2 + \nu(s)^2\right)ds$$

$$B_1(t,T) = \frac{1 - e^{-\kappa_r(T-t)}}{\kappa_r}$$

$$B_2(t,T) = \frac{\kappa_r\left(1-e^{\kappa_\varepsilon(t-T)}\right)-\kappa_\varepsilon\left(1-e^{\kappa_r(t-T)}\right)}{\kappa_r\kappa_\varepsilon(\kappa_r-\kappa_\varepsilon)}$$

The inclusion of the second factor has the effect of making the term structure movements more realistic than those predicted by the original Vasicek model. The HW model also allows humped volatility structures. Finally, by making θ a deterministic function of time, this model is able to match any initial term structure.

5.3.4 *Heath-Jarrow-Morton (HJM) Model*

This model proposes a general framework for arbitrage-free term structure modeling in continuous time. As a multi-factor model of the whole yield curve, it is broader in scope and more complex than the models discussed above; in fact, Puhle contends that all other interest rate models, including short rate models, can be expressed as special cases of this model.

HJM starts with the observed term structure of interest rates in its instantaneous forward rate form. For a fixed (but arbitrary) maturity $T \in [t, \tau]$, the following condition holds:

$$df(t,T) = m(t,T)dt + \sum_{i=1}^{d} s_i(t,T)dz_i(t)$$

Where, $m(t,T)$ = *drift,* $s_i(t, T)$ = i^{th} *volatility of* $f(t, T)$

$(z_1, \ldots, z_d)$ = *independent standard Brownian motions*

In its most general form, both drift and volatility depend on the entire forward rate curve at time t and the whole term structure of interest rates is modeled. Since the model has only finite Brownian motions as risk sources, an arbitrage condition for the forward rate drift – called the *HJM drift condition* – is specified (as below). It ensures an arbitrage-free market and implies that the drift cannot be chosen independently, but results from the specification of the volatility structure of the forward rates and the market prices of risk.

$$m(t,T)=\sum_{i=1}^{d}s_i(t,T)\left[\left(\int_t^T s_i(t,u)du\right)-\lambda_i(t)\right]$$

The short rate is considered the instantaneous forward rate with instantaneous maturity and is obtained by taking the limit $T \to t$ of the instantaneous forward rate function. Finally, the basic relationship between bond prices and interest rates is given by,

$$P(t,T)=e^{-\int_t^T f(t,u)du}$$

A stochastic discount factor approach is used to price the securities in an arbitrage-free market and the following expression for ZCB prices is obtained,

$$P(t,T)=E_t\left[\exp\left(-\int_t^T f(u,u)du+\sum_{i=1}^{d}\int_t^T \lambda_i(u)dz_i(u)-\sum_{i=1}^{d}\frac{1}{2}\int_t^T \lambda_i(u)^2du\right)\right]$$

VI. TERM STRUCTURE ESTIMATION

The term structure models introduced above suggest structures for the yield curve. However, many practical applications of these models – especially those involving pricing and hedging interest-rate derivatives – require estimates of the term structure in order to calibrate the model parameters in the first place. Term structure estimation is thus a necessary step in testing these models; it also provides an empirical alternative to the theoretical approach followed by dynamic term structure models.

The need for estimation stems from the fact that at any point in time, the whole term structure is not directly observable. While government bonds are observed to establish an interest rate benchmark, several confounding factors exist[21]: (1) The number and maturity of observable bonds is finite, while the term structure is over a continuum; (2) All bonds with maturity > 1 year (except those created using the STRIPS program) are coupon-bearing, while term structure models are designed to price ZCBs; (3) Term structure models do not account for practical issues such as default, callability, liquidity and/or tax privileges.

The term structure of ZCBs is of particular interest since it provides the short-term interest rate, which is used as a state variable by most term structure models. The first work dealing with the extraction of the (unobserved) ZCB term structure was that of McCulloch (1971); various other approaches have been proposed since. We limit our discussion to three methods: McCulloch (cubic splines), Fama/Bliss (1987) and Extended Nelson-Siegel (1996).

6.1 Bootstrapping

Perhaps the easiest way to determine the term structure of interest rates is to choose a representative series of bonds and obtain their yields for different maturities from the quoted market prices via interpolation. The U.S. Treasury yield curve described earlier is constructed in this manner. Yield curves can also be built from the prices of cash in the money market – a commonly used technique is to use LIBOR rates for the short end of the curve (t ≤ 3 mo.), futures to determine the mid-range of the curve (3 mo. ≤ t ≤ 15 mo.) and interest rate swaps to determine the long end of the curve (1 yr ≤ t ≤ 60 yr). Despite being no more than a quick and coarse approximation, bootstrapping is widely used by practitioners to determine the forward rates.

6.2 Notation

The following notation is commonly used in term structure estimation: the prices of N bonds, observed at the same point in time, are taken as given and denoted P_i ($i = 1 \ldots N$). These bonds provide known cash flows c_{ij} at times t_{ij} ($t_{ij} = 1 \ldots T_i$) in the future, where c_{ij} is the cash flow from bond i at time t_{ij}. The discount factor, denoted $d(t_{ij})$, is the discounted present value of one dollar to be received at time t_{ij}; it is obtained from the static no-arbitrage condition where the market price of each bond is the present value of its CFs, i.e.,

$$P_i = PV(c_i) = \sum_{t=1}^{T} c_{ij} \times d(t_{ij})$$

In practice, small deviations from the above (predicted) price are inevitable. For one, we observe quoted bid/ask prices, not the actual trade price – mismatches in the timing of price quotes and trades could easily result in small errors. In addition, other considerations like liquidity and taxes could result in certain risk premiums being reflected in the price. The following modification is therefore made,

$$P_i = PV(c_i) + \varepsilon_i$$

Where, ε_i = random error term

6.3 Fama-Bliss Method (1987)

This method – a form of bootstrapping – considers the term structure in terms of the forward rate curve *f(t)*, which is defined by the following relation and assumed to be constant between successive observed bond maturities,

$$d(t) = e^{-\int_0^t f(v)dv}$$

Let the sequence of observed bonds be ordered by increasing maturity, with T_i being the maturity of the i^{th} bond. F_i is the constant forward rate on the interval $(T_{i-1}, T_i]$ where $T_0 = 0$, i.e., $F(T) = F_i$ for $T \in (T_{i-1}, T_i]$. The discount function now becomes,

$$d(T) = \exp\left[-F_K(T - T_{K-1}) - \sum_{k=1}^{K-1} F_k(T_k - T_{k-1})\right]$$

Where, k is chosen such that $T \in (T_{k-1}, T_k]$

To bootstrap the forward rate curve, the value of $d(T_1)$ computed thus is used in the bond price expression for P_1 (provided in Section 6.2) to solve for F_1, and so forth. By construction, this technique exactly prices all the in-sample bonds; it is, however, subject to errors if some 'mis-priced' bonds are included in the sample. To mitigate the effects of this, Fama/Bliss propose a series of filters to eliminate disagreeable quotes.

6.4 McCulloch Cubic Spline Method (1975)

This method uses a cubic spline to estimate the discount function. Using the formulation of Jeffrey et al. (2000), the estimated discount function is of the form,

$$\hat{d}(t) = g_i(t) \text{ on interval } [t_i, t_{i+1}] \text{ for } i = (1, \ldots, k-1)$$

Where, $t_1, \ldots, t_k$ *are a pre-specified set of knot-points where* $t_1 = 0$

$$g_i(t) = a_i(t - t_i)^3 + b_i(t - t_i)^2 + c_i(t - t_i) + d_i$$

$$g_i^{1/n}\left(t_{i+1}\right)=g_{i+1}^{1/n}\left(t_{i+1}\right) \qquad \textit{for } i=1,\dots,k-2 \textit{ and } n=0,1,2\ (n^{th}\ \textit{derivative}),\ \ g_1(0)=1$$

This structure implies that for *k* knot points, there are *k+1* free parameters, which can be solved for using the standard ordinary least squares objective function, i.e,

$$\min_{\theta}\sum_{i=1}^{N}\varepsilon_i^2$$

Where, θ = set of free parameters in cubic spline

While providing a reasonably accurate spot rate curve, cubic splines – especially those with a large number of nodes – tend to oscillate. As Waggoner (1997)[38] points out, large oscillations are undesirable, particularly at longer maturities. This is because in a risk-neutral world where interest-rate curves contain expectations of future prices, large oscillations can imply large variations in these expected prices. Several methods have since been proposed to mitigate this problem, the most popular being regression splines, exponential splines and smoothed splines.

6.5 Extended Nelson-Siegel Method (1996)

This method, proposed by Bliss (1996) is a modification of the original Nelson-Siegel approximation (1987)[32]. According to the author, it brings together several desirable characteristics – it accounts for bid-ask spreads, fits the discount-rate function directly to bond prices and uses the asymptotically flat approximating function,

$$r(m)=\beta_0+\beta_1\left[\frac{1-e^{-m/\tau_1}}{m/\tau_1}\right]+\beta_2\left[\frac{1-e^{-m/\tau_1}}{m/\tau_1}-e^{-m/\tau_2}\right]$$

Where, $\beta_0, \beta_1, \beta_2$ = regression coefficients, m = maturity,

τ_1, τ_2 = time constants associated with the equation

The parameters $\Phi = [\beta_0, \beta_1, \beta_2, \tau_1, \tau_2]$ are then estimated using the following non-linear optimization estimation procedure,

$$\min_{\phi} \sum_{i=1}^{N_t} \left(w_i \cdot \varepsilon_i \right)^2$$

Where,

$$\varepsilon_i = \begin{cases} P_i^A - \widehat{P}_i \ \textit{if} \ \widehat{P}_i > P_i^A \\ P_i^B - \widehat{P}_i \ \textit{if} \ \widehat{P}_i < P_i^B \\ 0 \qquad \textit{otherwise} \end{cases}$$

$$\textit{Weights}, \ w_i = \frac{\left(1/d_i \right)}{\sum_{j=1}^{N_t} \left(1/d_j \right)}$$

d_i = *Macaulay duration (in days)*

Subject to:

$$r(m_{\min}) \geq 0$$

$$r(\infty) \geq 0$$

$$\exp\left[-r(m_k) m_k \right] \geq \exp\left[-r(m_{k+1}) m_{k+1} \right] \qquad \forall m_k < m_{\max}$$

VII. A BOND PORTFOLIO OPTIMIZATION MODEL

The techniques described in previous sections can be synthesized to form a portfolio optimization framework tailored to bonds. The general approach is to choose a particular term structure model, use it to obtain the expected returns and covariance matrix of returns for the bonds and then use these values as inputs in the traditional Markowitz approach to obtain the optimal portfolio. Given that this process involves subjective choices, several different approaches have been proposed. Three of these – Puhle, Korn/Koziol and Bazayit – were studied, of which the latter is presented here.

The Bazayit approach uses the Nelson-Siegel term structure estimation method to obtain the state variable for the Vasicek yield curve model. He considers only ZCBs and assumes continuous compounding. These assumptions result in the following specification of the spot rates,

$$R(t,T) = -\frac{\log P(t,T)}{T-t}$$

Where, $R(t, T)$ = yield of ZCBs on period $[t, T]$

The Nelson-Siegel yield function is given by,

$$R(T) = \beta_0 + \left(\beta_1 + \beta_2\right)\left[\frac{1-e^{-T/t}}{T/t}\right] - \beta_2 e^{-T/t}$$

Where, $B = [\beta_0, \beta_1, \beta_2, \tau]$ = regression parameters

The parameters are estimated by minimizing the sum of squared errors (SSE) of the Nelson-Siegel yield function, i.e.,

$$\min_B \sum_{\theta=1}^{m} \left(R(T) - \hat{R}(T)\right)^2$$

Where, $R(\theta)$ = observed bond yield of time to maturity θ for a given day

$\hat{R}(\theta)$ = theoretical bond yield of time to maturity θ for that day

The constraints will be specific to the particular bonds chosen. For instance, the author (separately) uses each of the following sets of initial conditions and constraints in his empirical study:

(1) $B_{initial} = [R_{long}, R_{short} - R_{long}, 1, 50], \quad \beta_0 > 0, \quad \beta_0 + \beta_1 = R_{short}$
(2) $B_{initial} = [R_{long}, R_{short} - R_{long}, 0.9, 49], \quad \beta_0 > 0, \quad \beta_0 + \beta_1 = R_{short}$
(3) $B_{initial} = [R_{long}, R_{short} - R_{long}, 1, 50], \quad \beta_0 > 0, \quad \beta_0 + \beta_1 < R_{short}$
(4) $B_{initial} = [R_{long}, R_{short} - R_{long}, 0.9, 49], \quad \beta_0 > 0, \quad \beta_0 + \beta_1 < R_{short}$
(5) $B_{initial} = [R_{long}, R_{short} - R_{long}, -1, 50], \quad \beta_0 > 0, \quad \beta_0 + \beta_1 < R_{short}, \quad \beta_2 < 0, \quad 25 \le \tau \le 800$

Where, R_{long} = *yield of bond with longest maturity on any given day*
R_{short} = *yield of bond with shortest maturity on any given day*

The Vasicek yield function (presented in Section 5.3.1) may be expressed as $R(T,r(t))$ – a smooth function of T and a stochastic process of t whose evolution is predicted by the Vasicek short rate process; it is used for future estimation of the yield curve. The minimization process involves forming a parameter set $P = [\kappa, R(\infty), \sigma_r, r]$ and using initial values $\hat{P} = [\hat{\kappa}, \hat{R}(\infty), \hat{\sigma}_r, \hat{r}]$ provided by the term structure estimation to approximate the yield curve. In his empirical study, Bazayit uses Monte Carlo simulation to obtain the mean estimate of $R(T)$, with 1000 trials conducted for each day.

Once the term structure has been estimated, the expected returns (μ_i) and covariance of returns ($\sigma_{i,j}^2$) of all bonds $i, j \in (1,\ldots,N)$ over the period $t = (0 \rightarrow T)$ may be determined. In a discrete scenario, they would be given by:

$$\mu_i = \frac{E_0\left(P\left(T,T_i\right)\right)}{P\left(0,T_i\right)} - 1 \quad , \quad \sigma_{i,j}^2 = Cov_0\left(\frac{P\left(T,T_i\right)}{P\left(0,T_i\right)}, \frac{P\left(T,T_j\right)}{P\left(0,T_j\right)}\right)$$

As seen above, the Vasicek model can have one or more stochastic factors. To determine the above moments, the moments of the factors are needed. The expectation and variance of the k^{th} factor at planning horizon $t = T$ are properties of the O-U process and are given by,

$$E_0\left(X_k(T)\right) = X_k(0) \cdot e^{-\kappa_k T}$$

$$Var_0\left(X_k(T)\right) = \frac{\sigma_k^2}{2\kappa_k}\left(1 - e^{-2\kappa_k T}\right)$$

The expected returns and covariance matrix may now be calculated. Empirical studies by Korn/Koziol (see Section 9.4) suggest that these estimates are most accurate (relative to the realized risk-return profile) in case of two-factor models.

VIII. PROFESSIONAL BOND PORTFOLIO STRATEGIES

Despite the availability of sophisticated term structure models, advanced optimization techniques and powerful computational tools, the most widely-used strategies in bond portfolio management remain relatively straightforward, time-tested and readily implementable. These strategies, which may be broadly classified into passive and active strategies, are described in this section.

8.1 Passive Management Strategies

Passive management strategies do not attempt to 'beat the market' by utilizing asymmetric information to achieve superior asset allocation or market timing; rather, they take prices as fairly set and attempt to manage portfolio risk. Two main strategies of this type are seen in the fixed-income markets – indexing and immunization.

8.1.1 Indexing

Indexing strategies attempt to create portfolios whose composition (and thus performance) closely matches that of an index or benchmark. Whereas the S&P500 is the most popular stock index, the three major indices of the bond markets are the Lehman (now Barclays Capital) Aggregate Bond Index, Salomon Smith Barney Broad Investment Grade (BIG) Index and Merrill Lynch U.S. Broad Market Index. All three are market-value-weighted indices of total returns and are comprised of government, corporate, mortgage-backed and Yankee bonds.

Bond indexing has seen a surge in popularity in recent years. Mossavar-Rahmani (1991)[30] attributes this to the poor historical performance of active bond managers. Fabozzi/Fong reveal that indexed portfolios also tend to have lower management and non-management fees, thereby passing on a greater portion of any gains to investors.

There are two major decisions to be made in pursuing an indexing strategy: the choice of an appropriate index and the creation of the index fund portfolio. The former is based on the investor's objectives, risk tolerance and any restrictions (imposed either by investors or regulators) on the type of securities which may be held. The portfolio construction process is in itself an interesting challenge – most bond indices contain over 5000 securities, many of which are quite illiquid and difficult to acquire at fair value. Further, the fixed maturity of bonds, issuance of new bonds and periodic coupon payments (which need to be reinvested) all necessitate frequent rebalancing, which can be tedious and expensive. This makes matching the index quite difficult.

In practice, it is deemed infeasible to precisely replicate the broad bond indices. A *stratified sampling* approach is commonly used, with bonds grouped by maturity, issuing sector, coupon rate, credit risk, etc. and a few bonds from each group held in proportion to the group's share of the overall market. As with equity portfolios, additional holdings (beyond a certain point) bring diminishing returns; in other words, most of the diversification benefit may be achieved by holding relatively few bonds. McEnally/Boardman (1979)[28] examine the impact of portfolio size on diversification. They present the following expression for variance of expected returns of a portfolio consisting of *n* randomly selected issues held in equal proportions (*1/n*),

$$Var(P_n) = Var(M) + \frac{1}{n}\left[Var(R_1)\right]$$

Where, n *= no. of securities in portfolio* $Var(P_n)$ *= variance of portfolio exp. returns*

$Var(M)$ *= average systematic variance of returns of bond universe*

$Var(R_i)$ *= average unsystematic (diversifiable) variance of one-security portfolios from which the n securities are drawn*

The ultimate goal remains to match the performance of the broad index. Any discrepancy between the returns of our portfolio (called the *indexed portfolio*) and that of the index is referred to as the *Tracking Error*. It has 3 sources: (1) Transaction costs in constructing the indexed portfolio; (2) Differences in the composition of the indexed portfolio and the index; (3) Discrepancies between the prices used by the index and those actually available to the index portfolio manager. The tradeoff between tracking error and number of issues in the indexed portfolio should be self-evident – the more closely the indexed portfolio matches the composition of the index, the less the tracking error and vice versa.

Fabozzi/Fong present an alternative to the equally-weighted approach used by McEnally/Boardman. Their method consists of the following steps:

- Define the classes into which the index universe is to be divided. Make the number of classes = number of securities to be held.
- Select securities to hold.
- Use quadratic programming to calculate the security weights. The objective is to minimize the sum of squares of the differences between the relative weights in the portfolio and the weights in the class. The constraints are (1) Match the duration of the portfolio and the index; (2) Match other characteristics.

In conclusion, it is worth pointing out that indexing is by no means a risk-free strategy. It does not eliminate risk; it merely seeks to matches the risk-reward profile of its target index.

8.1.2 Immunization

Immunization is the practice of shielding portfolios from interest rate risk. It is widely used by institutions such as pension funds, insurance companies, etc. which need to fund specific liabilities. The need for immunization arises from the natural mismatch between the maturities of institutions' assets and liabilities. For example, commercial bank assets typically consist of commercial/consumer loans outstanding, while their liabilities are primarily customer deposits. Clearly, the assets have longer maturities (and thus longer durations) than the liabilities, making them more sensitive to interest rate fluctuations. Immunization strives to match the interest rate exposures of assets and liabilities, thereby ensuring that declines in the value of one are offset by the appreciation of the other.

One way of achieving this is by *Duration Matching*. Recall that interest rate risk consists of price risk (lower prices due to higher interest rates) and reinvestment rate risk (declining interest rates result in coupon payments being reinvested at lower rates). These risks counteract each other; when the durations of assets and liabilities are matched, they cancel each other out.

This does not imply that duration matching eliminates the need for rebalancing. The instruments chosen to obtain the initial duration match will invariably have different convexities; even if interest rates stay the same, the passage of time will cause the respective durations to change by unequal amounts, resulting in a mismatch. Therefore, immunized portfolios must be rebalanced to account for interest rate changes and the passage of time. The frequency of rebalancing depends on the tradeoff between transaction costs and the potential impact of mismatched durations, both of which will reduce expected return. Thus, while ostensibly a passive strategy, immunization is only passive to the extent that it does not involve searching for undervalued securities.

Classical immunization theory makes two important assumptions: (1) The yield curve only undergoes parallel shifts; (2) The investment horizon is fixed. Under these conditions, it establishes a floor for the portfolio value, ensuring that it always remains at or above its base value if interest rates remained constant. In practice, however, yield curve shifts are seldom parallel, so immunization remains an imperfect measure.

Several techniques have been proposed to mitigate this problem: one approach is to alter the definition of duration to accommodate non-parallel yield curve shifts (multifunctional duration). A second approach, proposed by Fong/Vasicek (1984)[15] establishes a measure of immunization risk against any arbitrary interest rate change. It defines immunization risk as the risk of reinvestment; intuitively, pure discount instruments maturing at the investment horizon will have no reinvestment risk and therefore zero immunization risk. The goal is thus to construct a portfolio that replicates the payoffs of an instrument of this type. The authors demonstrate that if forward rates change in an arbitrary (including non-parallel) manner, the relative change in portfolio value depends on the product of two terms. The first of these terms is a measure of immunization risk and depends solely on the structure of the portfolio, which makes it controllable. It is given by,

$$\frac{\sum_{t=1}^{N}\left[PV\left(CF_t\right)\times\left(t-H\right)^2\right]}{\textit{Initial Investment}} = \frac{\sum_{t=1}^{N}\left[\frac{\left(CF_t\right)}{\left(1+y\right)^t}\times\left(t-H\right)^2\right]}{\textit{Initial Investment}}$$

Where, *$PV(CF_t)$ = Present value of cash flow at time t* *y = Present YTM*

H = length of investment horizon *N = time to receipt of last portfolio CF*

The second term is uncertain and characterizes the nature of the interest rate shock. The objective function minimizes the immunization risk, subject to the constraint that the portfolio duration equals the investment horizon. A linear programming solution to the problem is proposed.

This general framework was later extended to accommodate multiple liabilities, yielding a technique called *Multiple Liability Immunization*. This method imposes the following necessary and sufficient conditions: (1) The composite duration of the portfolios must equal that of the liabilities; (2) The distribution of asset durations must have a wider range than the distribution of the liabilities. These conditions have significant practical implications – the former implies that immunizing a liability stream does not require the asset maturities to match the liability timelines (this is particularly useful for insurance firms which have liabilities into perpetuity, while the maturities of available bonds extend only around 30 years hence). The immunization risk term in the multiple-liability scenario then becomes,

$$\frac{\sum_{t=1}^{N}\left[PV\left(CF_t\right)\times\left(t-D\right)^2\right]-\sum_{t=1}^{M}\left[PV\left(L_t\right)\times\left(t-D\right)^2\right]}{Initial\ \ Investment}$$

Where, $PV(L_t)$ *= Present value of liability at time t,*

D = portfolio duration (= weighted average duration of liabilities)

M = time of the last liability payment

The optimal strategy is to minimize the immunization risk measure subject to the constraints imposed by the two conditions; as before, linear programming is used to solve the problem.

8.2 Active Management Strategies

As the name suggests, active management strategies actively attempt to beat the market. There are two sources of potential value in active bond management – interest rate anticipation and identification of relative mispricing.

8.2.1 Interest Rate Anticipation Strategies

This refers to a class of strategies based on managers' expectations of future interest rates across the entire fixed-income market. As mentioned previously, interest rate changes are the dominant source of marginal total returns; as long as interest rates remain volatile, active managers must anticipate the effect of future rate changes.

Three aspects of any interest rate change must be anticipated: (1) Direction of the change; (2) Magnitude of the change across maturities; (3) Timing of the change. Recall that duration is a measure of interest rate sensitivity, with bonds of longer durations being more sensitive to interest rate fluctuations. Therefore, responses to anticipated interest rate changes often address the portfolio duration – if a drop in rates is anticipated, the active manager will shift to bonds of longer duration (higher maturity or lower coupon rate) to maximize the expected price rise, and vice versa. The positioning of the portfolio on the maturity spectrum would be based on the shape of the anticipated change, while the timing of the change would be a factor in evaluating its relative importance. In practice, interest rate forecasting is very difficult and of questionable accuracy. Scenario analysis is typically used to frame the analysis, with probabilities assigned to each scenario based on the strength of the analyst's convictions on the quality of his forecasts.

Additional factors can be imposed to reflect the effect of quality, issuing sector or coupon group on the basic yield curve shift. For instance, in a declining rate environment, high quality issues may continue to be priced at a premium. Such factors - termed *volatility factors* – modify the price response of the issue to interest rate changes, based on the unique characteristics of the bond. Fabozzi/Fong present the following expression for the volatility factor for an issuing sector, based on simple linear regression of historical data,

$$\Delta S_t = \alpha + (1-\beta)\Delta T_t + \varepsilon_t$$

Where, ΔS_t =interest rate change for the issuing sector in month t (in basis points)

ΔT_t = interest rate change for the Treasury issue in month t (in basis points)

ε_t = error term in month t

α, β= regression parameters

(α = issuing sector spread change, β = issuing sector volatility)

8.2.2 *Identification of Relative Mispricing*

The bond price formulae presented above are used in conjunction with subjective beliefs about risk (as they relate to bond risk premia) to produce an estimate of fair value for a bond. Any deviation of the actual price from this estimate suggests possible mispricing, which might present an arbitrage opportunity. Active managers seek to identify and profit from such opportunities; Homer/Liebowitz (1972)[18] characterize portfolio rebalancing as one of the following swaps,

- Substitution Swap: If two bonds with identical (or near identical) characteristics (coupon rates, maturities, call features, credit quality, etc.) are priced differently, a manager might choose to sell the one with the lower YTM in lieu of the one with the higher YTM.
- Inter-market Spread Swap: This is pursued when the yield spread (the difference between YTMs) in two markets is believed to be out of line. For instance, if the spread between Treasuries and corporate bonds is considered too wide, the manager will move from Treasuries → corporates.
- Rate Anticipation Swap: Based on interest rate forecasting, discussed above.
- Pure Yield Pickup Swap: This is intended to increase holding-period-return (HPR) by moving into higher-yield securities. For instance, if the term structure is upward-sloping, the manager will move into longer-term bonds. As long as the yield-curve does not shift during the holding period, the investor will gain more as compensation for bearing the interest rate risk.
- Tax Swap: A swap between two similar bonds based on the desirable tax characteristics of one of them.

While these strategies might suggest relatively easy profits, there are several mitigating factors: (1) This approach relies heavily on the quality of the underlying analysis – faulty analysis can present opportunities where there are none; (2) Even if mispricing does exist, it is likely to disappear quickly as investors bid up the price of the cheap bonds; (3) There is no telling how much time it will take for the expected corrections to materialize; (4) Unforeseen events can cause adverse price movements, resulting in losses.

The last point is fascinating when viewed in a historical context. A classic case in point is the failure of Long Term Capital Management – a sequence of events that has entered Wall Street lore. LTCM was a hedge fund which devised fixed-income statistical arbitrage strategies (called *convergence trades*, since they typically took both sides of the trade) to make bets on relative mispricing in foreign markets, using leverage (as much as 25:1) to amplify their profits. One such trade involved betting on the closing of the spread between (cheap) Russian government bonds and U.S. Treasuries. The unexpected default on its debt by the Russian government in 1998 led to a massive (and rapid) flight to quality (towards U.S. Treasuries), which lost the firm $1.85 billion.

IX. EMPIRICAL STUDY

An empirical study is conducted to test some of the ideas presented above. It is designed to fulfill two broad objectives: (1) To establish the need for an explicit portfolio approach in fixed-income investing; (2) To assess the viability of the Markowitz method in fixed-income investing and compare its performance with those of other approaches. Details of the study are provided here; the emphasis is on the underlying rationale, methodology used and findings.

9.1 Interest Rate Volatility

It has been suggested that one of the factors contributing to the limited use of MPT in the bond markets was low historical interest rate volatility, which precluded the need for an explicit portfolio approach in fixed-income investing. This theory is investigated here by constructing a historical time series of interest rates and asking the following questions: Are interest rates more volatile now than in the past? How is this measured? What effect has this had on bond yields (and therefore, prices)? What were the correlations across issuing sectors? How were these changes correlated to the performance of equities?

The first step is to narrow down the definition of the term 'interest rates', since several different rates are quoted in the markets. A distinction is sometimes made between short and long-term interest rates; the most widely-used short-term rate is the *Federal Funds Rate* – the rate at which private depository institutions (typically banks) lend money held at the Federal Reserve to other depositary institutions (mostly overnight). Another popular short-term rate is the yield on a 3-month Treasury bill. Long-term rates are those quoted on issues of longer maturity (for instance, on a 10-year Treasury note).

This study examines the variation in both short and long-term rates. The following monthly data is obtained from the Federal Reserve Statistical Release H.15[15]: the Federal Funds Effective (FFE) Rate (Jul 1954–Oct 2009), Nominal

Yields on Treasury Constant Maturity (TCM) securities with maturities ranging from 1-month to 30-years (Apr 1953–Oct 2009), Yields on corporate bonds (Moody's seasoned AAA and BAA) (Jan 1919 – Oct 2009), Yields on Municipal Bonds (Bond Buyer Index) of 20-year maturity (Jan 1953–Oct 2009).

The following observations are made:

- The FFE rate increased dramatically in 1979-80, followed by a four-year period of highly volatile rates. This has been widely attributed to a change in the Fed's operating procedures, which saw them move from an interest rate target to a reserves-based target. The variation in FFE over time is shown in Figure 5.
- A general upward trend in interest rates – both short and long-term – is observed until this time, followed by a period of declines. This is true across all issuing sectors and maturities. This may be observed in Figure 6, which depicts the yields on 1-year TCM, 10-year TCM, AAA corporate, BAA corporate and Municipal bonds over the study period.
- Nowak (1991)[33] suggests a variety of statistics which may be used to measure the volatility of interest rates: the standard deviation of rates (σ), the coefficient of variation ($c_v = \sigma/\mu$), the absolute average percent change (AAPC), etc. The most widely used measure is the standard deviation of rates.
- The volatility of the FFE rate – measured by the trailing twelve month (TTM) standard deviation of rates – is shown in Figure 7. While the heightened volatility of the '70s and early '80s is clearly observable, no long-term volatility trend is observed. This agrees with the findings of Nowak (1991), but is at odds with those of Rosenblum/Strongin (1983)[35], who found the trailing 13-week standard deviation to be systematically higher after 1980. This may be explained by the fact that: (1) They use the Federal Funds Target (FFT) rate,

while the FFE rate is used here; (2) They use weekly rates (monthly rates are used here).

- Long-term interest rate volatility is represented by the TTM 12-month standard deviation of the yields on 10-year TCMs, shown in Figure 8. A consistently higher level of volatility is observed since 1984.
- While there is some disagreement on the volatility trends of short-term interest rates, it is widely acknowledged that long-term rates have grown more volatile since the 1980s.
- While it can be difficult to establish correlations between interest rate changes and equity market movements, these linkages have become easier to identify in the recent past. For the period from 1954-2009, the correlation between the FFE rate and the S&P500 index was calculated to be -0.2613; however, a much closer correlation (ρ = 0.8375) is observed from 1999 onwards. This can be confirmed visually from Figure 5.

Figure 5 **Federal Funds Effective (FFE) Rate vs. S&P 500, 1954-2009**

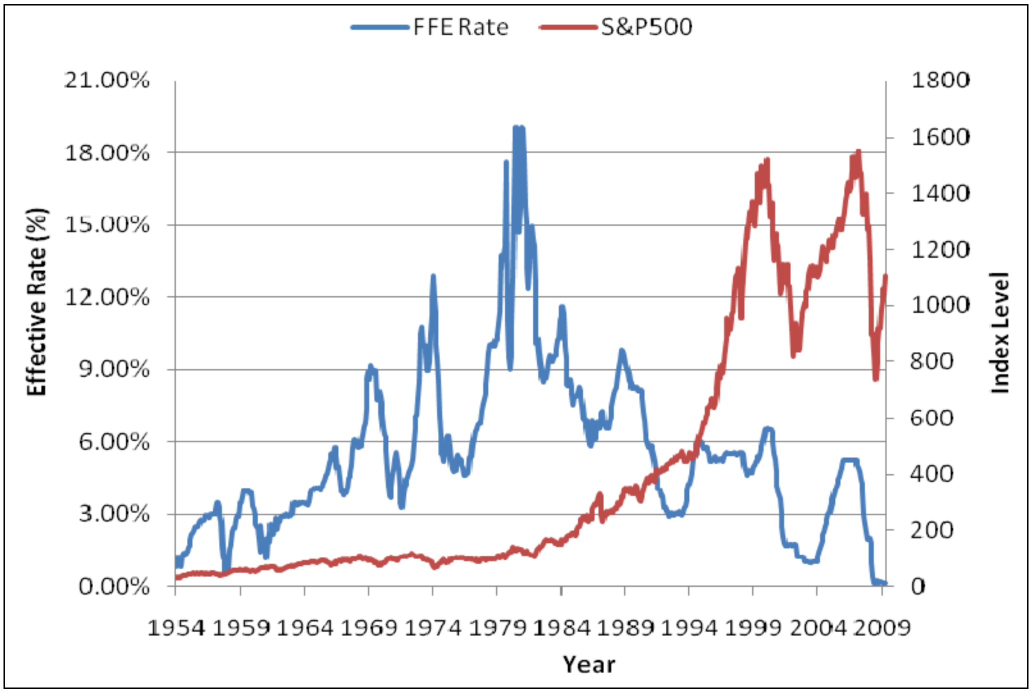


Figure 6 **Representative Bond Yields, 1953-2009**

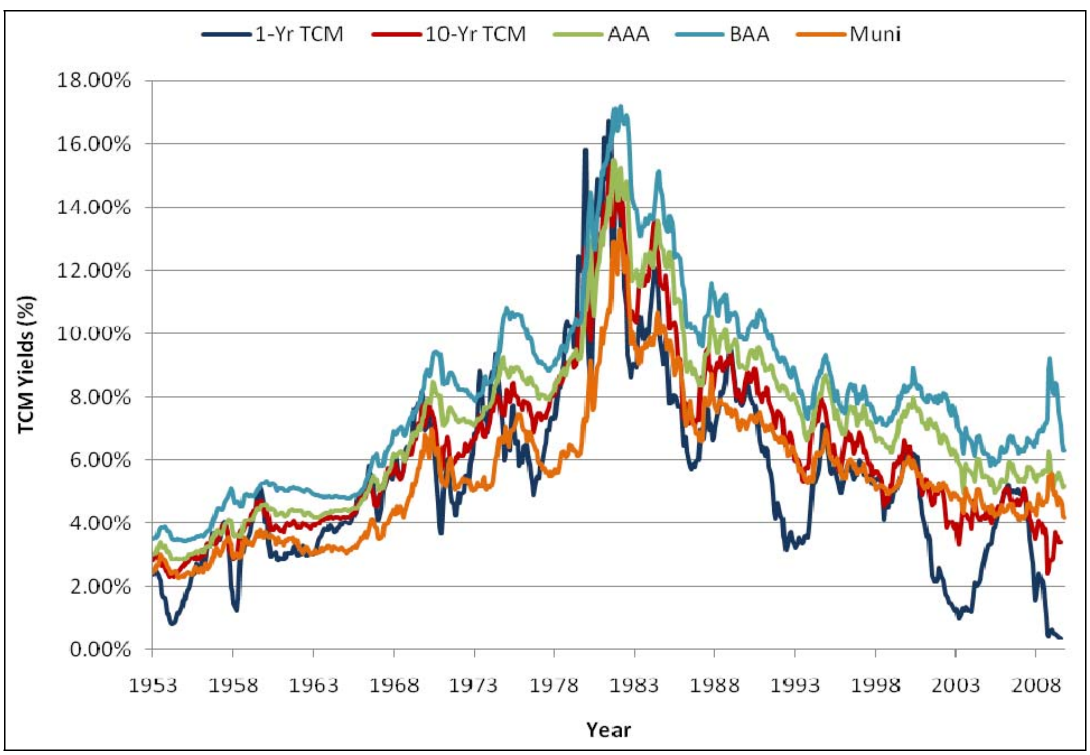


Figure 7 **Volatility of FFE Rate, 1954-2009**

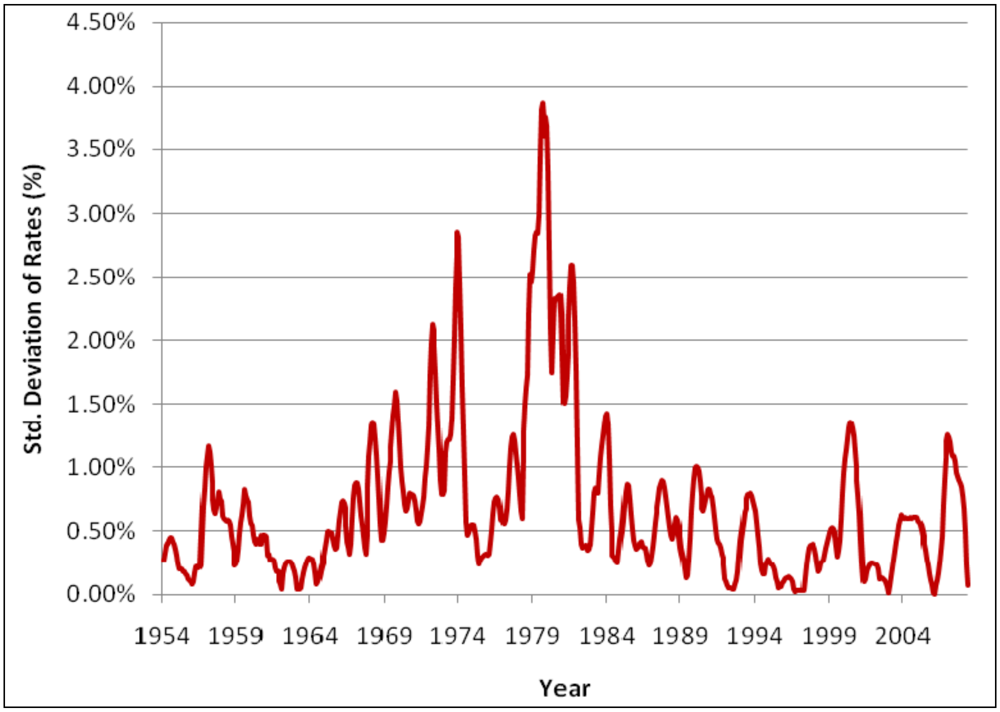
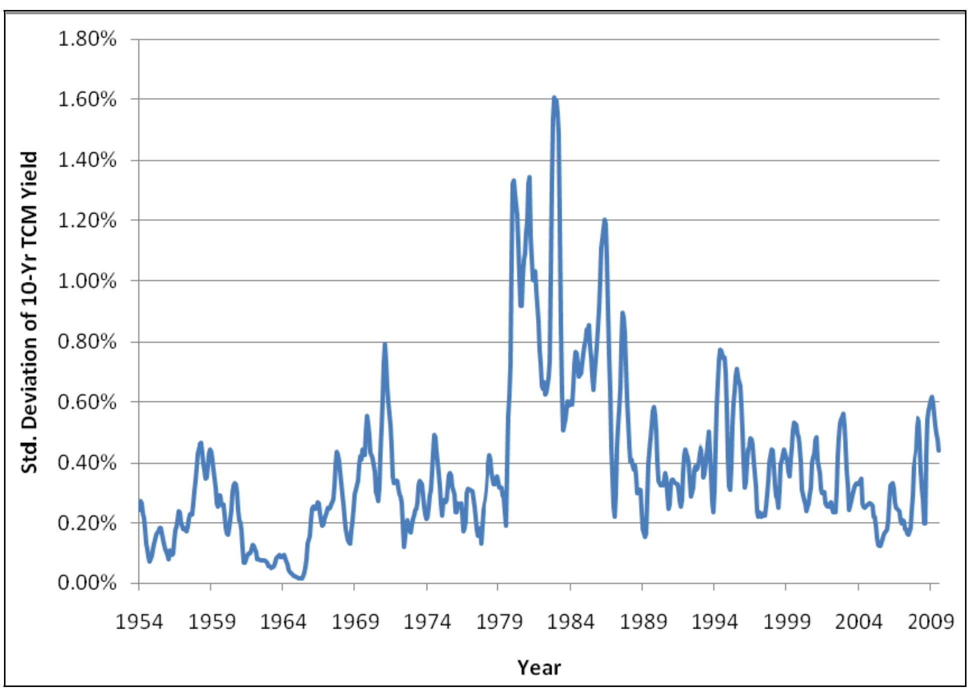


Figure 8 **Volatility of 10-year TCM Yield, 1954-2009**



9.2 Bond Indexing

Looking beyond the fundamental question of whether to choose active or passive investing, even committed passive investors must decide which specific technique to use. If the goal is investment (as opposed to the avoidance of risk) they will choose to purchase bonds, either individually or as part of a portfolio. The difficulties inherent in bond portfolio construction make it tempting to consider an indexing strategy; while this presents challenges of its own, the recent profusion of bond index funds and ETFs has greatly facilitated this process. This part of the study examines the relative merits of bond indexing.

As described previously, bond indexing strategies attempt to replicate the performance of a benchmark portfolio comprised of a representative set of bonds. It is worth noting that index performance never translates completely to investment returns, since all index funds charge investors a small management fee to cover their costs. For this part of the study, two indices and one index fund are chosen. The two broadest indices covering the U.S. bond markets – the Barclays Capital (formerly Lehman Brothers) U.S. Aggregate Bond Index and the J.P. Morgan (formerly Bear Stearns) U.S. Aggregate Bond Index are chosen as benchmarks. Since they are both notional constructs, a tradable index-fund based on the former is chosen as well – Barclays Capital U.S. Aggregate Bond Fund (NYSE: AGG). It is important to understand the difference between *indices* and *index funds* – it is impossible to invest in an index; any investments designed to mimic index returns must be made in index funds. Since AGG has only been in existence since 2003, a six year study period from Sep ’03–Aug ’09 is chosen. Performance statistics of the two indices, the index fund and the 10-year Treasury note over this period are presented in Table 1. Notice that the fund returns are slightly lower than the index it tracks – this is due to the management and other fees mentioned above.

Table 1 **Bond Index Performance, Sep '03 – Aug '09**

Index/Fund	*Avg. Monthly Return*	*Avg. Annualized Return*	*Cumulative Return (Study Period)*
J.P.Morgan U.S. Aggregate Index [a]	0.43%	5.36%	29.83%
Barclays Capital U. S. Aggregate Bond Index [b]	0.39%	4.99%	27.56%
Barclays U. S. Aggregate Bond Fund [c]	0.34%	4.28%	23.32%
10-Year Treasury Note	0.41%	5.04%	27.90%

[a] Based on Bloomberg data
[b], [c] *Source*: Barclays Capital/iShares

AGG is one of the most liquid and well-diversified bond index funds available. As the above results illustrate, this in itself is no guarantee of superior returns (over a single bond); however, any returns from the latter approach would be highly dependent on the characteristics of the bond chosen and the market conditions over the holding period. Diversification is desirable over a relatively long investment horizon, especially when the intent is to use a purely passive investment strategy. While an index fund like AGG does bear additional default risk (compared to Treasuries), it is sufficiently well diversified to mitigate the impact of defaults in all but the most adverse economic circumstances. Therefore, AGG is chosen as the investable benchmark for this study.

9.3 Portfolio Optimization of Bond Index Funds

The theoretical shortcomings of the Markowitz method in dealing with bond portfolios are well known. But what if this method were applied to portfolios of bond funds? Funds trade in much the same manner as equities; holding them would provide exposure to fixed-income and avoid the maturity and cash-flow reinvestment related problems posed by bonds. An added benefit is the ability to invest in actively managed funds, while ourselves following a passive management strategy.

The easiest way invest in bond funds would be to hold a single broad bond fund like AGG. However, it is also possible to construct a portfolio of targeted bond funds, tailored to our specific investment goals and risk tolerance. This study investigates the merits of this approach - the Markowitz method is used to construct an optimal portfolio of funds, with techniques like stratified sampling used to refine the portfolio construction process. A five year study period from Nov 1, 2004 – Nov 2, 2009 is chosen; the portfolio is held over the entire period (with no rebalancing) and its returns compared with those of the index (AGG).

9.3.1 Portfolio Construction

The first step is to select the funds to include in the portfolio. A variety of targeted funds are available to choose from, each providing exposure to certain parts of the bond markets. The following broad guidelines are followed in the portfolio construction process:

- Fund Diversification: Diversity is sought across issuing sector and maturity. A stratified sampling approach is used to help structure the universe of bond funds according to these characteristics. Not all aspects of the method (as originally designed) are used here – not all cells in the grid need be represented and holdings are not market-weighted.
- Eligibility: A fund is required to have existed for a minimum of 5 years prior to the start of the study period, so as to provide a reliable stream of historical data on which the analysis may be based.
- Fund Selection Criteria: Since portfolio performance is evaluated over a study period in the past, historical fund performance statistics are deliberately ignored in order to avoid any selection bias. Three main selection criteria are

used: nature of fund (active/passive), target risk/reward profile (government, investment grade, high yield, etc.), expense ratio (the ratio of expenses to total assets held).

- Short Sales and Inverse Funds: The concept of short selling is less prevalent in the bond markets than in equities. Also, while it is possible to invest in inverse funds (which target the inverse of the benchmark returns), it is unusual to hold such funds over an extended time period. Therefore, short sales and inverse funds are excluded from this study.
- Position Limits: There is no compelling case for pre-defined position limits. However, given the sensitivity of bonds to macroeconomic factors and since no rebalancing is allowed, it might make sense to impose position limits on some funds. This will be considered over the course of this study on a need basis.

Under these conditions, the funds listed in Table 2 are chosen for the portfolio. Exposure to different durations, issuing sectors and markets is achieved with a manageable sample of 5 bond funds, as shown in Table 3.

9.3.2 *Fund Parameter Estimation*

The Markowitz method uses the expected returns and covariances of returns as the inputs to the optimization model. While the covariance matrix is readily determined from the historical time series of returns, the expected returns are harder to estimate. The traditional approach (used in the case of equities) is to obtain them from the Capital Asset Pricing Model (CAPM):

$$E(R_i) = r_f + \beta_i \left[E(R_M) - r_f \right]$$

Where, E(R) = expected returns of security i *$E(R_M)$ = expected market returns*

β_i = parameter relating security i returns to that of the market

r_f = risk free rate of return

Table 2 **Bond Fund Portfolio – Funds Selected**

Fund Name	*Ticker*	*Type*	*Target Sector*	*Inception Date* [a]	*Expense Ratio* [b]	*Morningstar Rating* [c][d]
Vanguard Short-Term Bond Index	VBISX	Mutual[e] – Index	All	Apr 30, 1994	0.22%	★★★★★
Vanguard Interim-Term Investment Grade	VFICX	Mutual - Active	All – Inv. Grade	Dec 31, 1993	0.26%	★★★★
Legg Mason WA Muni High Income A	STXAX	Mutual – Active	Municipal	Nov 6, 1992	0.77%	★★★★
PIMCO High Yield Institutional	PHIYX	Mutual - Active	Corporate – High Yield	Dec 15, 1992	0.55%	★★★★
PIMCO Emerging Markets Bond Fund	PEBIX	Mutual - Active	All	Jul 31, 1997	0.83%	★★★★

[a], [b], [c] *Source:* Morningstar
[d] Morningstar ratings are on a 1-5 star scale. Provided for reference; not used in fund selection.
[e] Mutual funds issue securities to investors, either directly or through intermediaries; ETFs are traded on an exchange.

Table 3 **Stratified Sampling – Portfolio Exposure Map**

ISSUING SECTOR	*MATURITY* [a]		
	Short	***Intermediate***	***Long***
U.S. Government	VBISX ●●● [b]	VFICX ●	IEF ●●●
U.S. Municipal/Agency/ABS	VBISX ●	VFICX ● PHIYX ● STXAX ●●●	
U.S. Corporate – Investment Grade	VBISX ●●	VFICX ●●●	
U.S. Corporate – High Yield	VBISX ●	VFICX ●●● PHIYX ●●●	
Emerging Markets	VBISX ●	VFICX ●● PHIYX ●●●	PEBIX ●●●

[a] Maturity: Short < 3 years, Intermediate 3-10 years, Long > 10 years
[b] Level of exposure: ● Some (<10%) ●● Moderate (11-40%) ●●● Significant (>40%)

Table 4 **Security Risk-Return Parameters**

Fund	*Beta* [a]	*Historical Yield Spread* [b]	*Expected Annual Returns, $E(R_i)$* [c]	*Std. Deviation of Expected Returns $\sigma(R_i)$*
VBISX	0.55	0.53%	4.31%	2.51%
VFICX	1.28	0.80%	4.54%	4.79%
STXAX	0.79	1.57%	4.80%	2.93%
PHIYX	0.78	4.12%	7.57%	7.56%
PEBIX	1.52	1.86%	5.03%	11.77%

[a] 10-year betas from Google Finance. Not used in calculations, provided for reference.
[b] Calculated from historical yield spreads provided by Hull, et al (2005)
[c] E(R) values calculated using historical yield spreads; adjusted for expenses.

Applying the CAPM expression to bonds is not straightforward, for two reasons. First, consensus beta values are somewhat elusive and highly sensitive to market cycles. To illustrate, consider the case of VBISX: Morningstar provides two betas for this fund: 0.52 (relative to AGG) and 1.02 (relative to its stated benchmark – Barclays Capital 1-5 Year Govt./Credit Index). Given that AGG is used as a proxy for the bond markets, a reasonable approach would be to use the beta relative to that index. However, the Morningstar betas do not specify the time period over which they are computed, which can be a problem since bond fund betas can change significantly over time. For instance, Google Finance reports the following betas for VBISX: 1-yr beta = 0.47, 3-yr beta = 1.02, 5-yr beta = 0.54, 10-yr beta = 0.55. All things considered, the 10-year betas are considered the most reliable; however, as will be explained below, they are not needed for this study.

Another complicating factor in applying the CAPM to bonds is the absence of a single estimate for the market risk premium. Bond risk premiums depend on market conditions and the riskiness of a particular bond; they vary widely across issuing sectors. This being the case, there are several possible ways to determine the appropriate risk premium for a particular security. They include the following:

- The historical bond market risk premium of 1.5% estimated by some studies may be used in the CAPM formula to determine *E(R)* directly.
- Historical risk premia for securities of a certain credit rating may be used. This study uses historical yield spreads provided by Hull/Predescu/White (2005)[19]. Yield spreads represent the difference between yields on bonds and long-term Treasuries. To calculate the yield spread on a bond fund, the fund's holdings in instruments of a certain credit rating are weighted by the reported yield spreads of that rating. Since historical yield spreads are a direct estimate of the risk premium on a security, using these values would require assuming a beta of 1. The only concern with this method is that the holdings information

is current (and thus potentially outdated). For funds whose composition changes drastically over time, this could lead to inaccurate risk premiums.

- A case can also be made for simply using the risk premium on the asset class most representative of a certain fund (usually the asset class that forms the majority holding of that fund). For instance, if a fund holds mostly AA assets, the risk premium for that asset class could be used for the fund.

All three methods are compared here. *E(R)* values are computed for each fund and adjusted for expenses. Given the inexact nature of these methods, simple linear regression is used to obtain the best fit with observed data. The minimum sum of squared errors (SSE) method is used to fit the calculated *E(R)* values to the following: (1) The realized annualized returns for each fund for the five-years prior to the study period (i.e., Dec '99 – Nov '04), (2) The realized annualized returns from 1999-2004, bootstrapped from the 10-yr and 5-yr total returns reported by Morningstar. In both cases, the weighted average approach (Method 2 above) is found to fit best. The full *E(R)* calculations may be found in Tables A1 & A2 (Appendix B). The calculated β and *E(R)* values are presented in Table 4 above.

9.3.3 Optimization Methodology and Results

Once the parameters have been estimated, the optimization process is fairly straightforward. Two methods are used – the Markowitz (Minimum Variance) method to determine the global minimum variance portfolio and the Expected Return method to determine the portfolio composition for target values of $E(R_p)$. The target values used range between 0% and 20% in steps of 1%. Only three portfolios are found to provide feasible solutions [$\Sigma w_i = 1$, $R_p = E(R_p)$] using the Expected Return method, those having $E(R_p)$ = 5%, 6% and 7%. The target weights for each case are presented in Table 5. As mentioned previously, short sales are not considered.

Table 5 **Portfolio Target Weights – No Short Sales**

Method [a]	$E(R_p)$	$\sigma(R_p)$	*Target Weights*				
			VBISX	*VFICX*	*STXAX*	*PHIYX*	*PEBIX*
M	5.17%	2.16%	0.5739	0.0000	0.3575	0.0686	0.0000
ER	5.00%	2.27%	0.4322	0.0000	0.4188	0.1490	0.0000
ER	6.00%	3.68%	0.0000	0.0000	0.5675	0.4325	0.0000
ER	7.00%	6.03%	0.0000	0.0000	0.2070	0.7930	0.0000

[a] M: Markowitz (Min. Variance) method, ER: Expected Return method

For each of these portfolios, the realized total returns over each year of the study period and the entire five-year period are determined and compared with those of the index (AGG). Since any description of returns is incomplete without a description of risk, the Sharpe ratio is calculated as well. It is given by:

$$Sharpe\ Ratio = \frac{R_p - r_f}{\sigma_P}$$

Where, R_p *= realized portfolio return*

The results are tabulated separately for the Markowitz and Expected Return methods in Table 6A and Table 6B, respectively. The minimum variance portfolio is found to outperform the index in three of the five years and over the aggregate study period (total returns 22.27% vs. 21.19%). It also has a significantly higher Sharpe Ratio (0.35 vs. 0.22), indicating a superior risk-reward profile (greater returns per unit risk). Each of the feasible Expected Return portfolios also outperforms the index in absolute terms, although in two of the cases, this comes at the expense of higher risk (as indicated by a lower Sharpe Ratio).

Table 6A **Study Results – Markowitz Method, No Short Sales**

Year	*Time Period*	*Realized Return*		*Sharpe Ratio*	
		Portfolio (R_p)	*Index (R_i)*	*Portfolio (R_p)*	*Index (R_i)*
1	Nov '04 – Nov '05	3.87%	1.91%	-0.11	-0.62
2	Nov '05 – Nov '06	7.18%	5.55%	3.37	0.58
3	Nov '06 – Nov '07	4.76%	5.79%	0.52	0.57
4	Nov '07 – Nov '08	-6.21%	-1.22%	-2.05	-0.96
5	Nov '08 – Nov '09	12.68%	9.15%	1.76	0.73
Study Period	Nov '04 – Nov '09	22.27%	21.19%	0.35	0.22

Index: Barclays U.S. Aggregate Bond Fund (AGG)

Table 6B **Study Results – Expected Return Method, No Short Sales**

Year	*Portfolio Realized Return*				*Sharpe Ratio*			
	P1 [a]	*P2*	*P3*	*Index*	*P1*	*P2*	*P3*	*Index*
1	4.47%	6.21%	5.50%	1.91%	0.36	1.09	0.45	-0.62
2	7.84%	9.74%	9.44%	5.55%	4.02	5.13	2.99	0.58
3	4.35%	3.22%	3.89%	5.79%	0.20	-0.25	-0.02	0.57
4	-9.72%	-21.28%	-28.11%	-1.22%	-2.16	-2.23	-2.25	-0.96
5	15.44%	24.92%	33.07%	9.15%	1.88	2.15	3.01	0.73
Study Period	22.20%	22.82%	23.80%	21.19%	0.29	0.18	0.15	0.22

[a] P1: Portfolio with $E(R_p)$ = 5% in ER method; P2: $E(R_p)$ = 6%; P3: $E(R_p)$ = 7%

Among all the portfolios considered, the minimum variance portfolio performs the best, with a realized return of **22.27%** and a Sharpe Ratio of **0.35**. The 5% $E(R_p)$ portfolio comes a close second, with a realized return of **22.20%** and a Sharpe Ratio of **0.29**. While the absolute returns of the 6% and 7% $E(R_p)$ portfolios are slightly higher, their Sharpe Ratios are much lower, which makes it the deciding factor.

The strong performance of the optimized portfolios is somewhat surprising. While some diversification benefit is to be expected, it would be hard to achieve any greater diversification than AGG; being a bond index fund, it is itself a portfolio of bonds. Further, the ability of a pseudo-passive portfolio (while the portfolio itself is passive, each of the funds within it is actively managed) to outperform an active one is quite revealing. The inferior performance of the portfolios in year 4 should come as no surprise – this period witnessed a sharp increase in bond default rates from a low of around 1% of all issues in 2007 to 4.4% by the end of 2008; active management is likely to have been quite useful in such a scenario. Oddly enough, the passive nature of the portfolio probably allowed it to benefit more from the recent resurgence in the markets – the portfolio outperformed the index the most in April and May '09, when the market turnaround happened. Whatever the reasons, this study does enough to suggest that Markowitz optimization with bond funds may be able to add value in fixed-income investing after all.

9.4 Notable Empirical Results in Literature

Finally, a couple of interesting empirical results from the literature are relevant to our study and thus worth a mention here.

In the context of our discussion of interest rates and their effect on bond prices (and thus yields), recall that the prevailing market interest rate is but one of the factors determining bond prices (albeit the most important one in most circumstances). An interesting question would be to ask how much of the variation in a bond's returns can be attributed to interest rate changes. This is addressed by Litterman/Scheinkman, who study the impact of rate changes on variation in ZBC Treasury returns. They propose a three-factor model to provide further granularity to the concept of interest rates, the factors being the *level* of interest rates, the *steepness* of the yield curve and the its *curvature.* Using econometric analysis to quantify the contribution of each of these factors, they come up with the remarkable insight that 98.4% of the variation in returns is accounted for by interest rate changes, with the level of the yield curve being by far the most important of the three factors. Their findings (reproduced in Table A3, Appendix C) have important practical implications, since they identify the primary risk factors to consider in designing interest-rate hedging strategies.

The second set of empirical results has to do with the application of term structure model-based portfolio optimization methods to bond investing. While replicating such studies is beyond the scope of this report, the results of others' work can be summarized here. Of the authors who have conducted studies of this nature, the study by Korn/Koziol stands out. They apply a Vasicek term structure model with initial parameters provided by MLE estimation (using a Kalman filter algorithm) to the German government bond market between 1974 and 2004. They show that an optimized portfolio performs better than simple benchmark strategies over this period, as long as not too many bonds and factors are used. Perhaps the most unique feature of their work is their study of the impact of the number of bonds and factors in the model – they find that a 2 factor model with 2 or 3 bonds provides the greatest benefit (highest Sharpe ratio) relative to benchmarks. Table A4 (Appendix C) summarizes their findings for a 2-factor model.

These studies, especially the latter, provide potential directions for future research; it is left to the interested reader to explore them further.

X. CONCLUSION

The fixed maturity and cash flow characteristics of bonds pose challenges to the application of mean-variance optimization to the fixed-income markets. The advent of term structure models, which facilitate the modeling of interest rate evolution over time, has eased some of these problems. Despite this, term structure model-based optimization techniques have been slow to be adopted by practitioners. This can be ascribed to several factors: (1) The sheer complexity of these techniques makes them difficult to implement and modify; (2) Conventional methods perform adequately in most circumstances, especially those involving specific needs like meeting deterministic future liabilities; (3) Some level of active management is inevitable due to the sensitivity of bond prices to interest rate changes. As a result, these methods find use only among the most sophisticated fixed-income investors.

The empirical study conducted reveals increased interest rate volatility over the past decade and a strong positive correlation between stocks and bonds; these factors emphasize the need for a portfolio approach in fixed-income investing. The advent of bond funds and ETFs provides a convenient alternative to holding bonds – their similarities with equities also mean that the Markowitz method can easily be adapted for use with these instruments. The study indicates that there might be potential benefits to constructing an optimal portfolio of bond funds and holding over the study period, as opposed to investing in a broad-based bond index fund or specific individual bonds. Finally, studies by other authors suggest that term structure model-based optimization techniques might provide superior returns to benchmark strategies.

APPENDIX-A Vasicek Term Structure Model - Derivation

This model, proposed by Vasicek (1977), assumes that the term structure of interest rates is completely determined by the current value of only one random variable - the short rate of interest. As described previously, the short rate is assumed to follow an Ornstein-Uhlenbeck process having SDE:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma_r dz(t)$$

Where, *$r(t)$ = short rate at time t,* *κ = speed of mean reversion ($\kappa > 0$),*
θ = (constant) mean reversion level (long-run stable mean of short rate)
σ_r = volatility of the short rate (constant)

The forward rate volatilities are assumed to be of the following form,

$$s_1(t,T) = s(t,T) = \sigma_r \cdot e^{-\kappa(T-t)}$$

The market price of interest rate risk is also considered to be constant, i.e.,

$$\lambda(t) = \lambda \qquad \forall t, t = (1, \ldots, T)$$

Where, λ = market price of risk

The initial instantaneous forward rate curve is given by,

$$f(0,T) = \theta + e^{-\kappa T}\left(f(0,0) - \theta\right) + \lambda \frac{\sigma_r}{\kappa}\left(1 - e^{-\kappa T}\right) - \frac{\sigma_r^2}{2\kappa^2}\left(1 - e^{-\kappa T}\right)^2$$

The short rate may be defined as the instantaneous forward rate with instantaneous maturity, i.e., it can be obtained by taking the limit $T \to t$ of the instantaneous forward rate function $f(t,T)$. With these specifications, the short rate is obtained to be:

$$r(T) = f(T,T) = f(t,T) + \int_t^T s(u,T)\left(\int_u^T s(u,s)ds - \lambda\right) \cdot du + \int_t^T s(u,T) \cdot dz(u)$$

$$\Rightarrow \quad r(T) = r(t)e^{-\kappa(T-t)} + \theta\left(1-e^{-\kappa(T-t)}\right) + \sigma_r \int_t^T e^{-\kappa(T-u)} dz(u)$$

This allows us to derive the zero coupon bond (ZCB) price expression as follows,

$$P(t,T) = E_t\left[\exp\left(-r(t)\frac{1-e^{-\kappa(T-t)}}{\kappa} - \theta\frac{\left(-1+e^{-\kappa(T-t)}+\kappa(T-t)\right)}{\kappa}\right.\right.$$
$$\left.\left. -\frac{1}{2}\lambda^2(T-t) + \frac{\sigma_r}{\kappa}\int_t^T \left(\left(e^{-\kappa(T-t)}-1\right)+\lambda\right)\cdot dz(s)\right)\right]$$

$$P(t,T) = \exp\left(-r(t)\frac{1-e^{-\kappa(T-t)}}{\kappa} - \theta\frac{\left(-1+e^{-\kappa(T-t)}+\kappa(T-t)\right)}{\kappa} - \frac{1}{2}\lambda^2(T-t)\right)$$
$$\times E_t \cdot \exp\left(\underbrace{\frac{\sigma_r}{\kappa}\int_t^T \left(\left(e^{-\kappa(T-t)}-1\right)+\lambda\right)\cdot dz(s)}_{x}\right)$$

Since x is a normally distributed random variable, it follows that

$$E[e^x] = \exp\left(E[x] + \frac{1}{2}Var(x)\right)$$
$$Var(e^x) = E[e^x]^2 \cdot \exp\left(Var(x)-1\right)$$

This leads to the following expression,

$$E_t \cdot \exp(x) = \exp\left[-\frac{\left(3+e^{-2(T-t)\kappa}-4e^{-(T-t)\kappa}\right)\sigma_r^2}{4\kappa^2}\right.$$
$$\left. + \frac{\left(1-e^{-(T-t)\kappa}\right)\lambda\sigma_r}{\kappa^2} + \frac{(T-t)\left(\sigma_r-\kappa\lambda\right)^2}{2\kappa^2}\right]$$

This can be substituted into the *P(t,T)* expression above and simplified to yield the ZCB pricing formula presented above,

$$P(t,T) = A(t,T) \times e^{-B(t,T)r(t)}$$

Where, $$B(t,T) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)$$

$$A(t,T) = R(\infty)\left(\frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) - \left(T - t\right)\right) - \frac{\sigma_r^2}{4\kappa^3}\left(1 - e^{-\kappa(T-t)}\right)^2$$

$$R(\infty) = \left(\theta + \lambda\frac{\sigma_r}{\kappa} - \frac{1}{2}\frac{\sigma_r^2}{\kappa^2}\right)$$

R(∞) = *Interest rate with infinite maturity*

APPENDIX-B Empirical Study – E(R) Calculations

<u>*Method 1*</u> Constant risk premium (1.5%) used with CAPM formula

Inputs: Risk-free Rate (r_f) = 4.0% $E(R_M) - r_f = 1.5\%$ Beta values from Table 4

E(R): VBISX - 5.28%, VFICX – 5.24%, STXAX – 4.73%, PHIYX – 4.95%, PEBIX – 4.67% (per annum)

<u>*Method 2*</u> Calculate weighted average risk premium based on historical values, use with CAPM

Table A1 **Expected Return Calculations**

Fund	*Risk Premia and Holdings by Credit Rating* [a]							*Weighted Average Risk Premium*	*Adjusted E(R)* [b]
	Aaa	*Aa*	*A*	*Baa*	*Ba*	*B*	*Lower*		
	0.40%	*0.47%*	*0.77%*	*1.43%*	*3.04%*	*5.42%*	*12.78%*		
VBISX	75.4%	5.7%	11.2%	7.8%	0.0%	0.0%	0.0%	0.53%	4.31%
VFICX	17.0%	17.6%	40.7%	21.0%	0.6%	0.1%	0.1%	0.80%	4.54%
STXAX	20.1%	2.9%	28.7%	31.8%	10.2%	4.1%	2.1%	1.57%	4.80%
PHIYX	0.0%	1.0%	11.0%	13.0%	40.0%	25.0%	10.0%	4.12%	7.57%
PEBIX	6.0%	1.0%	7.0%	58.0%	27.0%	0.0%	1.0%	1.86%	5.03%

[a] *Source:* Hull/Predescu/White (2005)
[b] Adjusted for expenses. Adjusted E(R) = E(R) – Expenses

<u>*Method 3*</u> Use risk premia corresponding to largest/representative holdings (shaded cells) in CAPM

E(R): VBISX – 4.18%, VFICX – 4.51%, STXAX – 4.00%, PHIYX – 6.49%, PEBIX – 4.60%

APPENDIX-B (contd.)

Table A2 **Regression Analysis of E(R) Calculation Methods**

Fund	*Expected Returns, E(R)*			*Observed Returns, R_o* [b]	*Reported Returns, R_r* [c]	*Squared Errors (Observed), $[E(R_i) - R_o]^2$*			*Squared Errors (Reported), $[E(R_i) - R_r]^2$*		
	M1[a]	*M2*	*M3*			*M1*	*M2*	*M3*	*M1*	*M2*	*M3*
VBISX	5.28%	4.31%	4.18%	4.98%	5.69%	0.0009%	0.0045%	0.0064%	0.0017%	0.0192%	0.0229%
VFICX	5.24%	4.54%	4.51%	6.18%	8.02%	0.0087%	0.0267%	0.0277%	0.0770%	0.1207%	0.1229%
STXAX	4.73%	4.80%	4.00%	4.34%	4.02%	0.0016%	0.0022%	0.0011%	0.0050%	0.0061%	0.0000%
PHIYX	4.95%	7.57%	6.49%	5.76%	6.88%	0.0066%	0.0328%	0.0053%	0.0371%	0.0049%	0.0015%
PEBIX	4.67%	5.03%	4.60%	12.73%	20.27%	0.6492%	0.5924%	0.6605%	2.4344%	2.3232%	2.4563%
					Sum of Squared Errors[d] →	0.6670%	0.6585%	0.7010%	2.5553%	2.4741%	2.6036%

[a] M1, M2, M3: Methods 1, 2, 3 from above
[b] Observed Returns: Annualized returns observed over 5-yr period from 12/1/99 to 11/1/04.
[c] Reported Returns: 5-yr annualized returns for 1999-2004, bootstrapped from Morningstar-reported 10-yr returns.
[d] Sum of squared errors (SSE) – Shaded cells indicate min SSE for each data set (observed, reported).

APPENDIX-C Empirical Results from Literature

Table A3 **Litterman/Scheinkman –Factors Driving Interest Rate Sensitivity**

Maturity	**Total Variance Explained**	**Proportion of Variance Explained By**		
		Level	***Steepness***	***Curvature***
6 months	99.5%	79.5%	17.2%	3.3%
1 year	99.4%	89.7%	10.1%	0.2%
2 years	98.2%	93.4%	2.4%	4.2%
5 years	98.8%	98.2%	1.1%	0.7%
8 years	98.7%	95.4%	4.6%	0.0%
10 years	98.8%	92.9%	6.9%	0.2%
14 years	98.4%	86.2%	11.5%	2.2%
18 years	95.3%	80.5%	14.3%	5.2%
Average	**98.4%**	**89.5%**	**8.5%**	**2.0%**

Source: Litterman/Scheinkman (1991)

Table A4 **Performance of Korn/Koziol Model vs. Benchmarks**

(Portfolio optimization method based on 2 factor Vasicek model)

# Risky Bonds in Model	**Return**	**Std. Deviation**	**Sharpe Ratio**
9	10.61%	18.42%	0.29
3	13.87%	20.96%	0.41
2	13.45%	19.72%	0.42
1	13.55%	11.44%	0.73
Benchmark 1 (DAX)	12.19%	25.75%	0.24
Benchmark 2 (REX)	12.75%	19.06%	0.40

Source: Korn/Koziol (2006)

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