FIXED INCOME SECURITIES

Valuation, Risk and Risk Management

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To Tommaso
Gabriele
and Sofia

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PREFACE

It is now the middle of 2009 and finally this book is completed. It has been very exciting to write a text on the risk and return of fixed income securities, and their derivatives, in the middle of what many consider the biggest financial crisis since the Great Depression. In these three years of work the world of finance changed, as many key players in fixed income markets either collapsed (e.g., the investment banks Bears Stearns and Lehman Brothers), have been acquired by the U.S. government (e.g., the two mortgage giant agencies Freddie Mac and Fannie Mae), or have been acquired by other banks (e.g., the investment bank Merrill Lynch). In this turmoil, the U.S. government has taken the center stage: On the one hand, the Federal Reserve decreased its reference short-term interest rate, the Federal Funds target rate, to almost zero, and acted swiftly to set up lending facilities to provide liquidity to the financial system. On the other hand, the U.S. Treasury used Congress-approved funds to bail out a number of financial institutions, while the Federal Deposit Insurance Corporation (FDIC) extended guarantees on the short-term debt of banks in risk of default.

What will this financial turmoil do to fixed income markets around the world?

While it is still hard to forecast how long the recession will last, a certain fact for now is that fixed income markets will get bigger. And this for several reasons: First, governments' debt will expand in the future, as governments across the globe increase their spending to stimulate demand and jump start their economies.[1] To do so, governments will need to borrow even more than in the past, thereby increasing government debt and thus affecting

[1]There is much disagreement on whether such fiscal stimulus will in fact work. However, there is little doubt that it will increase government debt.

the size of available fixed income securities. For instance, the U.S. government debt in marketable securities stood at about $6 trillion at the end of 2008, around 40% of the U.S. Gross Domestic Product (GDP), and the congressional budget office (CBO) predicts an additional $1.8 trillion U.S. deficit for 2009. A March, 2009 analysis of the CBO about the President's budget proposal even predicts an increase in U.S. debt held by the public to 56.8% of GDP by 2009 and up to 80% of GDP by 2019. [2]

Second, in 2008 the U.S. government took upon its shoulders the two mortgage giants Freddie Mac and Fannie Mae, and therefore their trillion dollars worth of debt can now be considered as safe (or as risky) as U.S. government securities, further expanding the effective size of U.S. government debt. The debt securities issued by these mortgage giants are not as simple as U.S. Treasury securities, as they have a number of additional features, such as embedded options of various kinds, that make their valuation and risk assessment difficult. The two agencies need to issue these types of securities to hedge against the variation in the value of the mortgage backed securities (MBS) that they hold in their assets, a variation that is mainly due to interest rate fluctuations. Indeed, the three agencies Fannie Mae, Freddie Mac, and Ginnie Mae hold or guarantee about half of the roughly $9 trillion U.S. mortgage market. This implies that about four to five trillion dollars worth of mortgage backed securities are now guaranteed by the U.S. government. It is only because of this guarantee, in fact, that the three agencies have been able to issue mortgage backed securities since the last quarter of 2008, whereas the private market completely dried up. Given the sheer size of the MBS market, it is as important as ever to understand the pricing and hedging of such complex fixed income securities.

More generically, a deep understanding of the forces that affect the valuation, risk, and return of fixed income securities and their derivatives has never been so important. Not only will fixed income markets be expanding in the future, as mentioned above, but in the past two years investors across the world dumped risky securities and purchased safe U.S. government securities, which pushed their prices up and their yields down. Understanding the forces that move the term structure of interest rates is important to determining what will happen to these prices once the crisis is over. For instance, how safe is an investment in long-term U.S. Treasuries? While the U.S. is (still) very unlikely to default, default risk is but one of the risks that affect the value of Treasury bonds, and an understanding of the possible losses from an investment in "safe" Treasury bonds is key, especially in an environment of low interest rates such as the current one. Indeed, the large expansionary monetary policy of the Federal Reserve, which was necessary to keep the banking sector from collapsing due to lack of liquidity, may spur a bout of inflation in the future. Inflation will affect the rate of return on nominal long-term bonds, and therefore the prices of fixed income securities will adjust. Is an investment in long-term U.S. Treasury bonds really safe? What about agency mortgage backed securities, which are guaranteed by the U.S. government as well? Is such an investment riskier than an investment in Treasury securities? How can derivatives help hedge risks?

[2] See *A Preliminary Analysis of the President's Budget and an Update of CBO's Budget and Economic Outlook*, Congressional Budget Office, March 2009.

About this Book

This book covers “fixed” income securities, their valuation, their risks, and the practice of risk management. I put quotation marks around the term “fixed” because nowadays most of the so-called fixed income securities have streams of income payments that are all but fixed. And it is exactly this fact, that most “fixed” income securities in modern financial markets actually do not have a “fixed income”, that makes the analysis of these debt instruments difficult. Let’s put some numbers down to see this more precisely. Consider once again the U.S. market: As of the end of 2008, the U.S. debt stood at about $6 trillion, approximately 90% of which is in Treasury securities that indeed have a fixed income, namely, with constant coupons that are paid over time. However, about 10% of the U.S. debt is in Treasury Inflation Protected Securities (TIPS), that pay a coupon that is not fixed at all, but fluctuates together with the realized U.S. inflation rate. These fluctuations make their valuation harder. On top of the $6 trillion Treasury debt, there is a $9 trillion mortgage backed securities market, whose securities (e.g., pass throughs, collateralized debt obligations, and so on) have streams of payments that are not fixed, but depend on various factors, including interest rates’ fluctuations. In addition, we should add the large swap market, now the main reference market for fixed income security dealers, which had a global market value of about $8 trillion in 2008. Once again, swaps do not have fixed income. And finally, the whole fixed income derivatives market, which includes forwards, futures and options, adds a few more trillion dollars.

What keeps these markets together?

The concept that I use throughout is the one of no arbitrage and the law of one price, that is, the fact that two securities that have the same cash flows should have the same price. In well-functioning markets, there shouldn’t be (large) arbitrage profits that are left on the table, as arbitrageurs would step in and trade them away. It is important to start from the no arbitrage principle to link all of these markets together. Then, after we have understood the concept of no arbitrage, we can look back and try to understand why sometimes apparent arbitrage opportunities seem to appear in the market, in the form of spreads between securities that look similar. Typically, the answer is risk, that is, it may be risky to set up an arbitrage strategy and carry it out. The 2007 - 2009 crisis provides in fact an important example of market disruptions, and this book contains several examples and case studies discussing the risk and return of setting up and carrying out what appear to be arbitrage strategies.

Why this Book?

The world of fixed income markets has being becoming increasingly more complex, with debt securities that have the most varied payoff structures, and fixed income derivatives that are growing in sheer size and complexity. Indeed, in many instances it is no longer clear what a real “derivative” security is. Typically, we think of a derivative security as a security whose value can be derived from the value of another, more primitive security from the rules of no arbitrage. However, when the size of a derivative market becomes larger than the one of the primitive securities, which price depends on which is not clear at all. The swap market, for instance, which we still call a derivative market and whose size at the beginning of the 1990s was negligible, now has a global market value of over $8 trillion, and a notional of over $350 trillion. While we can think of swaps as derivatives, in the more generic sense as hedging devices or non-funded financial instruments, their valuation does not derive from anything in particular, but only from the demand and supply of investors who use them for their needs to hedge or speculate in interest rates.

As the world of fixed income securities becomes more complex, I felt that anybody who studies fixed income securities must be exposed more directly to this complexity. This book provides a thorough discussion of these complex securities, the forces affecting their prices, their risks, and of the appropriate risk management practices. The idea here, however, is to provide a methodology, and not a shopping list. I do not go over all of the possible fixed income securities, structured products, and derivative securities that have ever been invented. I provide instead examples and methodologies that can be applied quite universally, once the basic concepts have been understood. For this reason, the book is filled with real-world examples and case studies, as discussed below. End-of-chapter exercises using real-world data and real-world securities cement the important concepts.

In addition, in modern financial markets, countries' central banks, such the Federal Reserve in the United States, actively intervene in fixed income markets to affect interest rates in the attempt to spur real growth and keep inflation low. A fixed income book cannot sidestep the central banks' influence on fixed income securities. I devote a chapter to discussing the Federal Reserve system, and the relation among interest rates, the real economy, and inflation. A large recent advance in academic literature links no arbitrage models with the activities of central banks, and this is important. Similarly, the academic literature has uncovered numerous stylized facts about the time variation of yields, which I also briefly summarize in a chapter. For instance, the old idea that an increasing yield curve predicts higher future interest rates has been proven false in the data time and again, and we should teach our students the implications of this empirical evidence. In particular, an increasing yield curve does not predict future higher rates, but future higher bond returns (i.e., if anything, lower future rates). That is, the literature has uncovered facts about the time variation of risk premia, which we should talk about in fixed income books. Without comprehending why yields move, students cannot have a complete understanding of fixed income markets.

The book also highlights the fact that most of the analysis of fixed income securities must rely on some models of the term structure, that is, some particular assumptions about the movement of yields through time. We use such models to link different types of instruments by no arbitrage and therefore establishing the price of one, perhaps complex, security by using the price of a more primitive security. Such models are used by market participants both to design arbitrage strategies in proprietary trading desks, or to value portfolios of derivatives for trading or accounting purposes, or to determine hedge ratios for risk management reasons. However, this book aims at clarifying two important issues: First, models have parameters and parameters need data to be estimated. Thus, the use of data and computers to determine models' parameters, and therefore to value fixed income securities, is just part of the fixed income game. We cannot propose to teach students even the basics of fixed income markets without a long and careful look at the data, and without knowing how to use data to fit models.

Second, the book clarifies that models are "just models", and they are always an incomplete description of a much more complex real world. We will see that different models may yield different answers about the value of the same derivative security even when using the same data to estimate their parameters. There isn't one right model. Each model has pros and cons and there is always a tradeoff between using one or another model. For instance, some models generate simple pricing formulas for relatively complex securities, and this simplicity is useful if a trader needs to compute the prices of a large portfolio of derivatives quickly. However, such models may be too simplistic to design an arbitrage strategy. More complex models take into account more features of the data, but they are also harder to implement. Finally, some models may work well in some type of interest rate environments, while others do not because of assumptions that must be made. In this book, we cover several models, and we go over their properties, the approximations of reality they make, and their possible drawbacks. The use of examples and case studies, as well as end-of-chapter exercises enables readers to grasp these differences and understand why one or another model may be useful in one or another circumstance.

Finally, my aim in writing this book was also to endow anybody who is interested in fixed income markets, even readers without a strong analytical background, to understand the complexities, the risks, and the risk management methodologies of real-world fixed income markets. With this desire in mind, I wrote the book in a way to cover all of the important concepts in each part of the book, as each part may require a different level of mathematical sophistication. Parts I and II of the book are accessible to students familiar with basic calculus, while Part III requires a more analytical background. Still, as discussed below, Parts I and II are sufficient to cover a complete course in fixed income, and they do cover all of the deep concepts that I believe anyone who studies fixed income and plays any role in these markets should possess. The world of fixed income securities has become more complex, and students who aim at working in this environment must now be able to recognise and work with this complexity.

I now describe the three parts of the book in more detail.

Part I: Basics

Part I of the book, Chapters 1 to 8, covers the basics of fixed income pricing, risk, and risk management. After introducing the main fixed income markets in Chapter 1, Chapter 2 contains the building blocks of fixed income relations, namely, the notion of discounts, interest rates, and the term structure of interest rates. The chapter also discusses the basic bond pricing formula, as well as some important methodologies for extracting discounts from observable bond prices. A case study at the end of the chapter further illustrates these concepts within the pricing of inverse floaters, which are popular fixed income securities yielding higher-than-market returns if interest rates decline.

Chapter 3 contains the basics of risk management: The chapter introduces the concept of duration, and its use to design effective hedging strategies, as in asset-liability management. The chapter also introduces the popular risk measures of Value-at-Risk and expected shortfall. The chapter illustrates these concepts with a discussion of the (likely) risks embedded in the portfolio of Orange County, which lost $1.6 billion and declared bankruptcy in 1994. Chapter 4 contains some refinements in the risk management techniques introduced in Chapter 3: In particular, the chapter illustrates the notion of bond convexity, and its implication for risk and risk management, as well as the concepts of yield curve's slope and curvature dynamics. This chapter shows that the notion of duration is an incomplete measure of risk, and relatively simple modifications to the model allow for a much better hedging performances, especially through the notion of factor neutrality.

Chapter 5 introduces basic interest rate derivatives, such as forwards and swaps. Besides describing their properties and their pricing methodology, several examples throughout the chapter also illustrate the use of such derivative contracts for an effective risk management strategy. The chapter ends with a case study discussing the risks embedded in a popular trade, a swap spread trade, a case that also provides further understanding of the swap market itself. Chapter 6 is the second introductory chapter on derivative securities, covering futures and options. In particular, the chapter illustrates the notion of options as financial insurance contracts, which pay only if some particular event takes place. After the description of futures and options contracts, several examples discuss the usefulness of these contracts for risk management. In addition, the chapter contains a discussion of the pros and cons of using forwards, futures, and options for hedging purposes.

A book on fixed income securities must mention the impact that monetary policy has on interest rates. Chapter 7 discusses the Federal Reserve policy rules, and covers in particular the Federal Funds rate. A case study at the conclusion of the chapter illustrates the activities of the Federal Reserve by using the financial crisis of 2007 - 2008 as an example. The chapter also introduces the Federal funds futures, and the information contained in such derivative contracts to predict future movements in the Federal funds rate. This chapter also connects the movement of interest rates over time to real economic growth and inflation rate, as the Federal Reserve acts to keep the economy growing and the inflation rate low. As the focus is on inflation, this chapter also covers the Treasury Inflation Protected Securities (TIPS), Treasury securities that pay coupons and principal that are linked to the realized inflation rate. Finally, this chapter contains the academic evidence about the variation over time of interest rates, and the fact that risk premia to hold bonds are time varying. In particular, this chapter answers the question of why the term structure of interest rates, on average, slopes upward.

The final chapter of Part I is Chapter 8, which contains a discussion of the mortgage backed securities (MBS) market, its main players, and the securitization process. Given that the financial market turmoil of 2007 - 2008 started in the mortgage backed securities markets, the chapter also describes some of the events during this period of time. This introductory chapter to mortgage backed securities also contains a discussion of the main measures of prepayment speed, as well as their impact on the pricing and risk exposure of several MBS, such as simple pass throughs, collateralized mortgage obligations, and principal only and interest only strips. The concept of negative convexity is thoroughly discussed, and illustrated by using data from the main trading market of agency pass throughs, the To-Be-Announced (TBA) market. A case study at the end of the chapter also demonstrates how we can measure the duration and convexity of MBSs (and other securities) by using data instead of pricing formulas.

Part II: Binomial Trees

The second part of the book introduces readers to the concept of term structure modeling and no arbitrage strategies. Chapter 9 illustrates these important concepts in the simple framework of one-step binomial trees. I use this chapter to discuss both the relative pricing of different fixed income instruments, the notion of risk premium of a fixed income security, as well as the popular pricing methodology called risk neutral pricing. The chapter does not use any more mathematics than Part I does, but it is the first step into a bigger world, the world of no arbitrage term structure models. Chapter 10 extends the analysis to multi-step trees. Students will learn the concepts of dynamic replication and hedging. These are strategies that allow a trader to hedge a contingent payoff in the future by using a portfolio of other fixed income securities, and understanding them is at the heart of no arbitrage pricing. The chapter also discusses a simple methodology to build long-term trees from the prediction of future short-term interest rates, as well as the concept of risk adjusted probabilities and risk premia. Real-world examples including the pricing of long-term structured derivatives illustrate how the methodology can be readily applied to price relatively complex securities. Finally, the chapter introduces the concept of spot rate duration, which is a concept of duration analogous to the one introduced in Chapter 3, but for securities defined on binomial trees.

Chapter 11 applies the concepts described in the previous two chapters to illustrate the no arbitrage pricing of numerous derivative securities. The chapter uses two popular models, the Ho-Lee model and Black, Derman, and Toy model, to show the differences in pricing between different models, even when the inputs are the same. These differences allow me to describe the various properties of the models. We use these models also to price standard derivatives, such as caps, floors, swaps and swaptions. In addition, the chapter introduces the notion of implied volatility, that is, the volatility of interest rates that is implied by the value of options. Building on these multi-step binomial tree models, Chapter 12 investigates the pricing of American options, that is, options that can be exercised any time before maturity. Several securities have embedded American options, including callable bonds and mortgage backed securities. This chapter illustrates the concepts of American options, and the methodology to price them, by going through several examples, such as Treasury callable securities, American swaptions, and mortgage backed securities. This chapter also shows the negative convexity that is generated by the American option feature embedded in such securities.

Finally, Chapter 13 illustrates a new methodology, Monte Carlo simulations, to price very complex securities on binomial trees. There are securities that cannot be easily priced on binomial trees because their payoff at maturity may depend on a particular path of interest rates. However, we can use computers to simulate interest paths *on the tree* itself, and therefore obtain the prices and hedge ratios of these securities by simulation. The chapter applies the methodology to relatively complicated real-world securities, such as amortizing swaps and mortgage backed securities.

Part III: Continuous Time Models

Part III covers more advanced term structure models that rely on continuous time mathematics. While this part is self contained, as it contains all of the important mathematical concepts, readers should be ready to see a substantial step up in the analytical requirement compared to the previous two parts of the book, which, as mentioned, instead only require a background in basic calculus.

Chapter 14 introduces the notions of Brownian motion, differential equations and Ito's lemma. I introduce the concept of a Brownian motion by relying on the intuition developed on binomial trees, namely, as a limiting behavior of rates as the time-step in the binomial tree converges to zero. Differential equations are introduced only through examples, as my aim here is to provide students with the notion of differential equations, and not the methodology to solve for them. I also illustrate the concept of Ito's lemma by relying on the convexity concepts discuss earlier in Chapter 4. I apply the concepts of Brownian motions and Ito's lemma in Chapter 15 to illustrate the notion of no arbitrage, and obtain the fundamental pricing equation, an equation that we can use to compute the price of any fixed income derivative. I focus on the Vasicek model, a model that is relatively simple but also realistic, and provide several examples on the pricing of real-world securities. In this chapter I tackle the issue of how to estimate the model's parameters, and show the potential shortcomings of the model. The chapter also illustrates the use of this model for the pricing of options.

Chapter 16 takes the model one step further, and discusses the issue of dynamic rebalancing and relative value trades. Essentially, all fixed income securities are linked to each other by the variation of interest rates, and therefore they move in a highly correlated fashion. An interest rate model allows us to compute the price of one security by using a portfolio of other securities, so long as the latter is properly rebalanced over time as interest rates change. The methodology is illustrated through various real-world examples, as well as a case study at the end of the chapter which features real data, and demonstrates the methodology in action. The chapter also illustrate some drawbacks of using simple models.

Chapter 17 introduces the second important result of continuous time finance, namely, the Feynman Kac formula, which provides the solution to the fundamental pricing equation obtained in Chapter 16. This formula is at the basis of the risk neutral pricing methodology widely used by market participants to price fixed income securities. In addition, this formula also justifies the use of some type of Monte Carlo simulations to price fixed income securities. The chapter provides numerous real-world examples, as well as a case study discussing the fair valuation of the leveraged swap between Bankers Trust and Procter & Gamble, which was a the center of a famous court case in 1994. Indeed, Chapter 18 covers the topics of risk measurement and risk management within continuous time models: In particular, I illustrate the notion of market price of risk, the fair compensation that a fixed income investor should expect to realize when he or she purchases a fixed income security, as well as Monte Carlo simulations for risk assessment. I illustrate the use of Monte Carlo simulations for risk assessment both in examples, as well as in a case study at the end of chapter. The chapter also includes an economic model of the term structure, which links the continuous time models illustrated in earlier chapters to the variation in expected inflation, and the compensation for risk that investors require to hold nominal securities when there is inflation.

Chapter 19 discusses no arbitrage models, which are models similar to the ones introduced in Chapter 11 on binomial trees, but in continuous time. The inputs of these models are the bond prices, and the outputs are the prices of derivative securities. The chapter offers several applications, and further highlights the pros and cons of different types of models. I carry on this discussion in Chapter 20, which illustrates the Black's formula to price standard derivatives, such as caps, floors, and swaptions. This chapter also links back to Chapter 11 in what concerns the notion of implied volatility. The chapter also discusses the important concepts of flat and forward volatility, as well as the dynamics of the term structure of volatility over time. These concepts are so important in modern financial markets that I decided to present this material in isolation from the previous chapters in Part III, so that the material in this chapter stands alone, and can also be used as a concluding chapter after Chapter 11.

Chapter 21 introduces a more recent pricing methodology, the forward risk neutral pricing methodology, as well as the more recent Heath, Jarrow, and Morton (HJM) model, and the Brace, Gatarek, and Musiela (BGM) model. Several applications show the usefulness of these new models to obtaining the price of even more complicated securities, although often by relying on Monte Carlo simulations.

I conclude this third part of the book, and the book itself, with Chapter 22, which extends the concepts developed in the previous chapters to the case in which the yield curve is driven by multiple factors. Luckily, the main concepts developed earlier readily extend to multifactor models. I show the additional flexibility offered by these multifactor models to price interesting additional structured notes and derivative securities, such as those that depend on multiple points of the term structure.

Pedagogical Strategy

This book employs a hands-on strategy to highlight the valuation, the risks, and the risk management of fixed income securities. The text is filled with real-world examples and case studies, which I use to show step by step the fair valuation of most securities, the return an investor should expect from an investment, and the riskiness of such an investment. I always use data to set up an example or to illustrate a concept, not only because it makes the lesson more relevant, but because it shows that we can actually tackle real-world valuation problems by studying the concepts illustrated in each chapter.

Examples

Each chapter contains many numerical examples illustrating the concepts introduced in the chapter. Sometimes I use examples to motivate new important concepts. As mentioned, such examples are always based on real data and therefore on real situations. Even so, examples are stripped down versions of much more complex problems, and I use such examples to illustrate one issue at a time.

Case Studies

The book contains several end-of-chapter case studies. These case studies apply the concepts developed in the chapter to more complex real-world situations. Such situations may involve the pricing of some structured derivatives, or their risk assessment using some measures of risk, or describe an arbitrage trading situation and the risk involved in carrying it out. Unlike the examples, which are tightly focused on the particular issue just being discussed in the chapter, a case study describes a situation and then carries out the whole analysis, although of course still within the topic discussed in the chapter. I use case studies also to show that we must often make many approximations when we apply relatively simple formulas or models to real-world data. That is, the world is much more complicated than the simple models or formulas would imply.

Not all chapters have case studies, as it depends on the topic of each chapter. If a chapter is too simple, for instance, because it is only introductory, then it is hard to apply the concept to a real-world situation, which tends to be complicated.

Data

The book relies heavily on real-world securities data. I use data to illustrate the examples in the body of the textbook as well as to discuss the case studies at the end of chapters. In addition, most of the exercises require some data analysis. These data are collected in spreadsheets, which are available with the textbook. The decision to rely foremost on the use of data as a pedagogical device springs from my beliefs that only by *doing* the analysis with real-world numbers can a student really understand not only the concepts illustrated in the particular chapter, but also the complexity of applying models to the real world. From the very beginning we will see that it is actually hard to apply the simple formulas of fixed income models, even the most elementary ones such as a present value formula, to real-world data. It is important for students to realize this fact early on, and it is this challenge that makes the study of fixed income markets so fascinating.

Exercises

Each chapter contains several exercises that cover the topics discussed, and highlights additional features of real-world fixed-income securities or trading methodologies. A solutions manual is available to instructors. The exercises are an integral part of the learning strategy: Most exercises are data driven and require the use of computers, either spreadsheets (for Parts I and II) or a programming software (for Part III). In modern financial markets computers are just a necessary part of the analysis toolbox. For instance, in Part I exercises require spreadsheets to compute the prices of complicated securities from simpler ones, or their duration and convexity. In Part II, the exercises require spreadsheet programs to build binomial trees that fit real-world fixed income securities, such as bonds, swaps and options. Moreover, in some chapters, the exercise requires students to carry out Monte Carlo simulations, on the binomial trees, to value real-world fixed income securities with embedded options, such as the Bermudan callable bonds of Freddie Mac.

In Part III, the exercises again rely on real-world data to fit more complex models of the term structure, and ask students to price relatively complex securities. In addition, exercises often require students to carry out a risk analysis, by computing hedge ratios or risk measures. The hands-on approach will make clear why practitioners use one model or another in the various circumstances: Students will experience firsthand the difficulties of dealing with data even when using relatively simple models.

Software

There are numerous examples in the book which use real-world data to illustrate the concepts discussed in each chapter. Together with the data sets in such examples, I also include all of the spreadsheets (for Part I and II) or computer codes (for Part III) that generate the results of the analysis in the numerical examples. These spreadsheets and computer codes should be used as a guide not only to better understand the examples themselves, but also to carry out a similar analysis in the end-of-chapter exercises.

For Instructors

The material in this book can be taught at two different levels: An introductory level and an advanced (but not very advanced) level.

Course I: Introduction to Fixed Income Securities

Parts I and II introduce basic analytical tools, and students familiar with basic calculus should be able to follow them relatively easily. This material covers a full semester fixed income course for both MBA or undergraduate students. Yet, notwithstanding the relative simplicity of these two parts of the book, the hands-on strategy, the real-world examples, the case studies, and the focus on real-world securities provide a strong foundation for the important concepts in fixed income asset pricing, from no arbitrage to risk premia, from duration to positive and negative convexity, from risk measurement to risk neutral pricing. Students at the end of the course will have the tools to tackle the proper analysis of real-world securities, assess their risk, and perform Monte Carlo simulations (on binomial trees) to value complex securities. These tools are very important to uncover the often hidden risks in some structured interest rate securities.

Pedagogically, the chapters' order already offers guidance on how to progress with the material. Each chapter's content often contains the seeds of concepts described in future chapters. For instance, in Chapter 1 I describe the repurchase agreement (repo) market, because in Chapter 2, which covers the present value formulas and the use of the law of one price, I can leverage on the repo market to describe how financial institutions actually carry out long-short strategies. Similarly, in Chapter 2 I describe floating rate notes, not only because I can then use this concept to illustrate the pricing of inverse floaters (a case study at the end of the same chapter), but also because in Chapter 5 I use the same concept to describe the pricing of swaps, which is the largest fixed income market by notional amount (about $350 trillion at the end of 2008). The chapters are highly interrelated and cross-reference each other, and therefore I believe it is pedagogically important to move forward chapter by chapter.

There is one final remark I want to make in regard to an introductory course in fixed income. Part II of the book, and especially Chapter 11, discusses the pricing and hedging of plain vanilla derivatives, such as caps, floors, swaps, and swaptions. This chapter also dicusses the concepts of implied volatility, flat volatility and forward volatility in the context of two specific models, the Ho-Lee model and the Black, Derman, and Toy model. This part should therefore be useful to link this material to the notion of implied volatility from the Black formula, the standard market formula used to *quote* standard derivatives. With this link in mind, I wrote Chapter 20 in Part III in a way that does not need any of the more advanced material in the earlier chapters of Part III. I just introduce the Black formula, and discuss the dynamics of implied volatilities over time, and the concept of flat and forward volatility. The formula is as difficult as the Black and Scholes formula for options on stocks, so depending on how advanced the students are, they may or may not find the material challenging.

Course II: An Advanced Course in Fixed Income Securities

The advanced course would make full use of Part II and Part III of the book. This is the course I regularly teach to the second year MBA students at the University of Chicago Booth School of Business, and it is also appropriate for students enrolled in master in finance programs. The prerequisites for my course include an investment course and a basic option course, although I often let students with a solid mathematical background to take the course without the prerequisites. I cover briefly the concepts in Chapters 1 to 6, which serve mainly to set the notation for the course. I then teach both binomial trees (Part II) and continuous time models (Part III), more or less in the progression described in the book. Indeed, Chapter 14 refers to Chapter 11 to introduce the notion of a Brownian motion as the limit of a binomial tree, as the step size becomes infinitesimal small. The key concepts that are explored in Part II are then also repeated in Part III, but by using continuous time methodologies. Students find it very useful to see the same concepts introduced in binomial trees repeated in a continuous time framework, as their intuition gets solidified, especially through the plentiful examples. However, the greater flexibility offered by the continuous time model enables me to discuss many more models which are not covered in binomial trees, even with many stochastic factors (in Chapter 22). Students tend to enjoy the comparison across models, and why some models work in some interest rate environemnts and not in some others. To this end, I give my students challenging, data-oriented homeworks to make them aware not only of the vast possibilities offered by fixed income term structure models, and their usefulness to price, hedge or implement a risk analysis of a given security, but also to have students realize the limitations of such fixed income models, and the fact that models need data for their effective application to the real world. My homeworks are always based on real-world securities that need to be priced, hedged or, more generically, analyzed, and I wrote most of the end-of-chapter exercises in this book with this aim in mind, namely, to have students analyze real-world securities by using the models discussed in each chapter. Sometimes the analysis require students to gather data from other sources available on public web sites, such as the LIBOR fixes available at the British Bankers' Association Web site, or the swap rates, available at the Federal Reserve Web site. The data analysis is an integral part of the book and the learning experience. In term of material, finally, my students also find it useful to connect the economic model discussed in Chapter 18 to the Vasicek model, discussed earlier, as well as to the evidence on expected return in Chapter 7, as they see the connections between risk, risk aversion, return, market price of risk, and, ultimately, pricing.

Conclusion

To conclude this brief introduction to the book, let me mention that I truly hope that this book will encourage readers and students to analyze fixed income markets in a very systematic way, always looking for the reason why some events occur, some trades seem possible, or some models may or may not work. I hope that my decision to have two full parts of the book requiring only a minimal analytical background will push readers to try to correctly assess the riskiness of complex fixed income securities, to see better what they are buying, and whether there is any reason why a security may appear to yield a higher-than-market return. Similarly, regulators may use the same tools to assess the fair valuation of complex securities, at least to first order, without needing a Ph.D. in mathematics or physics. In the same way nowadays it is much harder to understand how the engine of a car works, compared to the past, and mechanics need to have a better knowledge than in the past about these new engines, participants in fixed income markets, whether traders, risk managers, regulators and so on, cannot hope to use old tools to understand modern markets, as their complexity has just being increasing through time, and new tools are necessary. I hope this book will provide the tools, even to the less mathematically oriented reader, to understand the complexities of fixed income modern markets.

PIETRO VERONESI

Chicago
June, 2009

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P. V.

PART I

FIXED INCOME MARKETS

CHAPTER 1

AN INTRODUCTION TO FIXED INCOME MARKETS

1.1 INTRODUCTION

In the past two decades, fixed income markets have experienced an impressive growth, both in market value and in complexity. In the old days, until the end of the 1980s, fixed income markets were dominated by government debt securities, such as United States government Treasury bills, notes, and bonds. These securities were also relatively simple, as the U.S. government mainly issued bonds paying a fixed amount of money semi-annually. Although other governments, such as those of the United Kingdom and Italy, also experimented with other types of debt securities whose semi-annual payments were not fixed, but rather linked to a floating index, for instance, the inflation rate, such markets were relatively small. Thus, the U.S. government debt market was the main reference for global fixed income markets.

Today, however, the U.S. government debt is no longer the dominant fixed income market, not so much because the U.S. debt shrank over the past two decades, but rather because other fixed income markets rose substantially relative to U.S. debt and became the main reference for fixed income pricing. Table 1.1 provides a snapshot of the sizes of fixed income markets as of December 2008. The first block of markets comprises the traditional fixed income markets, including U.S. government debt securities, municipal bonds, federal agency securities and the money market. The total size of these debt markets is around $15 trillion. The next block shows the size of the mortgage backed securities and asset-backed securities markets. In particular, the mortgage backed securities market stands as a $8.9 trillion market, a good $3 trillion larger than U.S. debt.

Table 1.1 The Size of Fixed Income Markets: December 2008

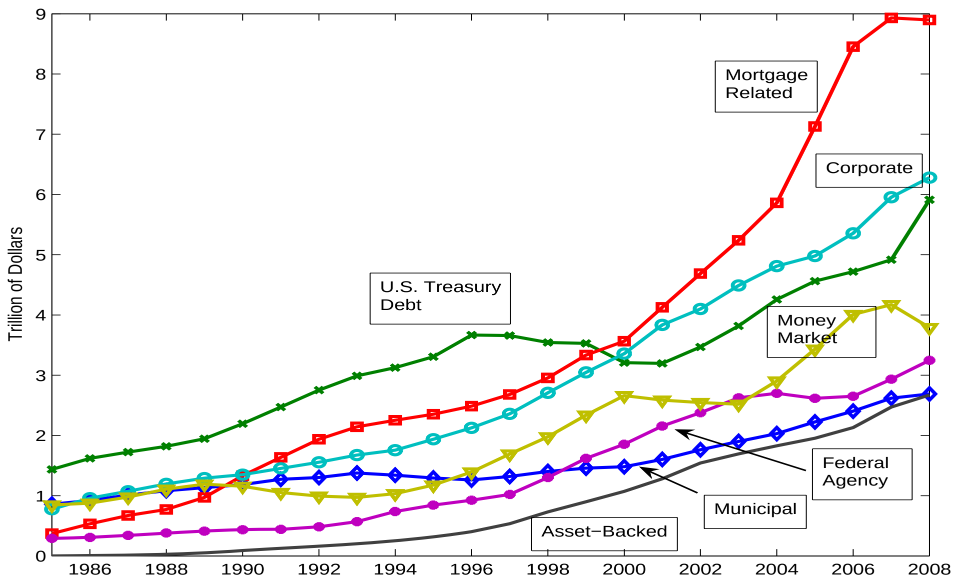
Market	Market Value (billion of dollars)	Notional (billion of dollars)
U.S. Treasury Debt	5,912.2	
U.S. Municipal Debt	2,690.1	
U.S. Federal Agency Securities	3,247.4	
U.S. Money Market	3,791.1	
Mortgage Backed Securities	8,897.3	
Asset-Backed Securities	2,671.8	
OTC Interest Rate Swaps	16,572.85	328,114.49
OTC Interest Rate Forwards	153.19	39,262.24
OTC Interest Rate Options	1,694.22	51,301.37
Exchange Traded Futures		19,271.05
Exchange Traded Options		35,161.34
U.S. Corporate Debt	6,280.6	
Credit Derivatives	5,651	41,868

Source: Securities Industry and Financial Market Association (SIFMA) and Bank for International Settlements (BIS).

Similarly, the next block of markets in Table 1.1 shows the interest rate derivatives markets. Interest rate swaps, in particular, have a market value of $16 trillion, and a notional value of $328 trillion. Although neither figure can be compared directly to the U.S. debt market, for a number of reasons discussed in Chapter 5, the sizes of these markets once again demonstrate that the U.S. debt market has been eclipsed by other types of securities. In particular, although in the 1980s and 1990s we would think of swaps as derivative securities, which "derive" their price from the value of primary securities, such as Treasuries, it is hard to believe that this is still the case now due to its sheer size. To any extent, we should consider the swap market a primary market whose value is driven by investors', speculators' and end users' fluctuating demands. Finally, corporate debt has increased dramatically in the past few years, with a debt value of about $6.2 trillion. Note too that the growing market of credit derivatives has reached a market value of $5.6 trillion, and a notional value of $42 trillion.

The changes in these markets are evident also in Figures 1.1 and 1.2. Considering first the Treasury debt market, we see that from 1986 to 1996 it grew steadily. The economic expansion that started in 1991, which would end in 2001, also generated a government surplus between 1996 and 1999, which led the U.S. government to initiate a policy of debt buyback. This is evident in the decrease in the face value of government debt during this period. The U.S. debt started growing again in 2001, to reach about $5.9 trillion in December 2008. The interesting fact about Figure 1.1, though, is the rise of another market, which has become a dominant market in the U.S., namely, the market of mortgage backed securities. From its value of only $372 billion in 1985 it increased steadily over time, to become larger than the U.S. debt market in 1999, and to become $3 trillion larger than the U.S. debt market by December 2008. The growth in this market is due to the growth of the U.S. real estate market, which boomed in the 2000's to reach its peak in 2006, as well as the steady increase in leverage of U.S. households, who had been taking larger

Figure 1.1 The Growth in Market Size



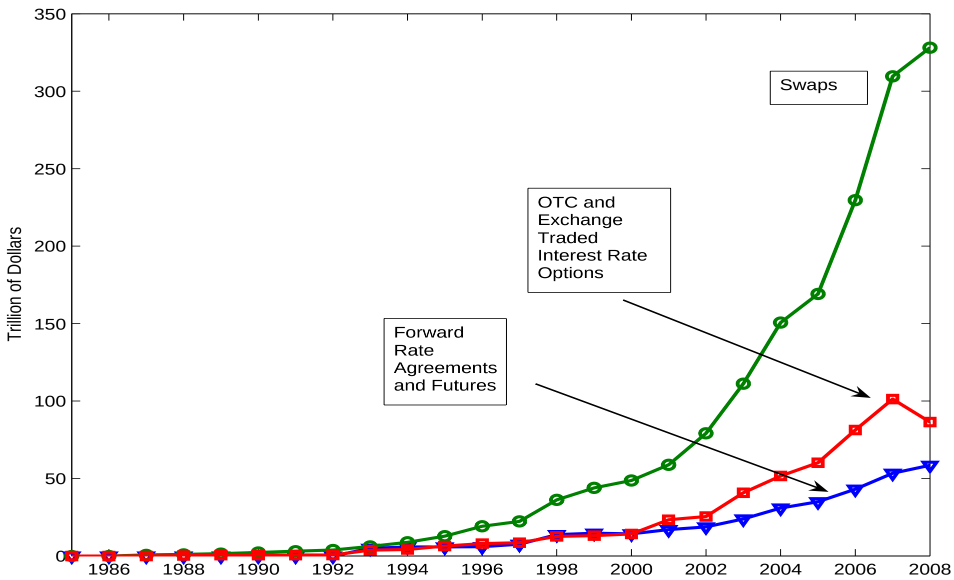
Source: The Securities Industry and Financial Markets Association (SIFMA)

and larger mortgages and home equity loans to finance consumption. The slight decrease in this market size visible at the end of the sample, in 2008, is indeed a reflection of the decline in the housing market and the U.S. recession that started in January 2007. Finally, a similar growth occurred in corporate debt, which next to U.S. debt was comparatively small in 1985, but grew steadily over the years, to reach $6.2 trillion by December 2008.

Figure 1.2 plots the stunning growth in interest rate derivatives markets. The interest rate swaps market, which was negligible at the beginning of the 1980s, experienced an exponential growth, reaching $328 trillion (notional) by December 2008. The figure also plots the combinations of over-the-counter (OTC) and exchange-traded interest rate options, which also grew considerably during this time frame, to reach about $100 trillion notional by December 2007, although it declined to $86 trillion by December 2008, in the midst of the 2007 - 2009 financial crisis. Similarly, forward rate agreements and futures contracts also grew over time, although at a much slower rate.

The bottom line of this discussion is that fixed income markets are very large and still growing. Moreover, there is not a dominant market: What we called a derivative market in the past is now larger in sheer size than the primary market. The big question is what is keeping the prices of the interest rate instruments tied to each other. That is, all of these instruments are highly correlated. For instance, if the Federal Reserve drops the Fed funds rate, then we may expect all of the short-term interest rates to fall. How do these rates move together? The answer is no arbitrage, that is, the possibility does not exist for arbitrageurs to take large positions in different securities whenever the prices across markets do not line up. The concept of “line up” will become clear in future chapters. For now, we turn to describing individual markets in more detail.

Figure 1.2 The Growth in Derivatives Markets: Notional



Source: SIFMA and Bank for International Settlement

1.1.1 The Complexity of Fixed Income Markets

The previous section illustrates the growth in size of fixed income markets. The complexity of fixed income markets is also extraordinary. Table 1.2 reports a snapshot of rates in the U.S. fixed income markets on September 18, 2007. The table corresponds to screen BTMM from Bloomberg terminals, and it is widely used by traders to quickly grasp the relative positions of bond prices and interest rates across markets. The number of securities described in this table is daunting. Starting from the top left corner, we have:

1. Federal funds rate quotes;
2. U.S. Treasury bill prices and yields at various maturities;
3. Eurodollar deposit rates at various maturities;
4. Repo and reverse repo rates;
5. U.S. Treasury bond yields and prices with various maturities;
6. Commercial paper quotes;
7. 90-day Eurodollar futures for various maturities;
8. Federal funds futures for various maturities;
9. LIBOR fixes;
10. Foreign exchange rates;

11. 30-years mortgage backed securities;
12. 10-year Treasury note futures;
13. Swap rates for various maturities;
14. Other key rates, such as the prime.

How do all these market rates move together?

The notion is that these quantities are all highly correlated with the same events. For instance, if there are worries of an increase in future inflation, we can expect the Federal Reserve to increase the target Fed Funds rate (see Chapter 7). In turn, this expectation as well as the rules of no arbitrage, discussed below, have an impact on other short-term borrowing rates, such as the short-term LIBOR, the short-term Eurodollar rates and so on. In this chapter we define only the terms appearing in Table 1.2. In the following chapters, we describe the relations between these markets and many others that do not appear in Table 1.2. The key concept is the concept of no arbitrage, which is helpful to introduce right away.

1.1.2 No Arbitrage and the Law of One Price

At the source of the ripple-through effect from one market to the next is the notion of no arbitrage. In its *pure form*, an arbitrage opportunity is defined as follows:

Definition 1.1 *An* **arbitrage opportunity** *is a feasible trading strategy involving two or more securities with either of the following characteristics:*

1. *It does not cost anything at initiation, and it generates a sure positive profit by a certain date in the future;*
2. *It generates a positive profit at initiation, and it has a sure nonnegative payoff by a certain date in the future.*

The **no arbitrage** *condition requires that no arbitrage opportunities exist.*

A pure arbitrage trade consists in taking positions that generate, magically to some extent, always nonnegative cash flows, and with certainty, some positive cash flow. There are three key elements in Definition 1.1: The trade (1) costs nothing; (2) yields positive profits with certainty; and (3) the profits arrive within a known time. For instance, if an arbitrageur finds two securities that pay exactly $100 in six months, but one trades at $P_1 = \$97$ and the other at $P_2 = \$98$, then an arbitrageur can apply the trader's motto "buy low and sell high," and purchase 1 million units of Security 1 at $97 and sell 1 million units of Security 2 at $98, realizing an inflow of $1 million. In six months, the two securities generate exactly the same cash flow and therefore the trader is hedged: Whatever he or she receives from Security 1 is then given to the holder of Security 2.

Of course, these types of pure arbitrage opportunities are hard to find in financial markets. Because of transaction costs and the lack of perfect co-movement among variables, some risks do in fact exist, and arbitrageurs must take them into account while they trade. The rules of no arbitrage, however, are still key to defining some relationships that must exist

Table 1.2 A Snapshot of U.S. Treasury and Money Market Rates

FED FUNDS		
BID/ASK	5 1/8	5 1/36
LST/OPEN	5 1/8	5 1/8
HIGH/LOW	5 1/8	5 1/8
DJIA	13479.91	+76.49

US T-BILL YIELD/PRICE				
4W	3.88	0.12	3.82	3.81
3M	4.16	0.02	4.07	4.17
6M	4.31	0.02	4.17	4.15
S&P 500 FUT		1947	+7.20	

EURO $ DEP		
3M	5.5000	5.6000
6M	5.3300	5.4300
1Y	5.0300	5.1300
CCMP	2592.02	+10.36

	REVERSE	REPO
O/N	5.15	5.05
1W	4.95	4.85
2W	4.85	4.75
1M	4.70	4.60

US BONDS YLD/BID/ASK/CHG				
4 08/31/09	4.117	99-24+	99-25	-03
4 1/2 05/15/10	4.128	100-29	100-29+	-04
4 1/8 08/31/12	4.241	99-15	99-15+	-05+
4 3/4 08/15/17	4.493	102-00	102-01	-07
5 05/15/37	4.735	104-05	104-06	-19+
		CRB	324.31	-.56

DEALER CP	
15D	5.060
30D	5.350
60D	5.400
90D	5.410
120D	5.380
180D	5.240

90D EUR $ FUT	
DEC	95.07
MAR	95.38
JUN	95.51
SEP	95.56
DEC	95.56
MAR	95.51 ‘

FUNDS FUT	
SEP	95.01
OCT	95.11
NOV	95.30
DEC	95.39
JAN	95.44
FEB	95.56

LIBOR FIX	
1W	5.25875
1M	5.49625
2M	5.55375
3M	5.58750
4M	5.53625
5M	5.48313
6M	5.42000
1Y	5.11250

SPOT FOREX	
JPY	115.8000
EUR	1.3875
GBP	1.9983
CHF	1.1876
MXN	11.1030
CAD	1.0233

30Y MBS			
GNMN 6.0	100-24	100-25	-02
GOLD 6.0	100-09	100-10	00
FNMA 6.0	100-07	100-08	-01

10yr Note Fut		
CBT	109 - 20	-04+
CRUDE OIL		
NYM WTI	80.80	+.23

SWAP RATES	
3Y	4.809
5Y	4.918
10Y	5.174

Key Rates	
Prime	8.25
BLR	7.00
FDTR	5.25
Discount	5.75

Source: Bloomberg. Screen BTMM. Date: September 18, 2007.

across assets prices, which in turn determine the relative prices of fixed income instruments. In this book we will see how these rules of no arbitrage allow us to both compute the fair value of fixed income instruments and to investigate their relative prices. Just as important, we will focus on the impact that no arbitrage has on the *risk*of fixed income instruments and therefore their *risk management*. At the basis of much of the analysis is the law of one price, discussed next:

Fact 1.1 *The* **law of one price** *establishes that securities with identical payoffs should have the same price.*

If the law of one price does not hold for some securities, then an arbitrage opportunity exists. Indeed, the logic is the same as the one of the previous example: if two securities have same cash flows in the future but trade at different prices today, then an arbitrageur could buy the underpriced security and sell the overpriced one, realizing a profit today. Since the cash flows are the same in the future, the arbitrageur is perfectly hedged.

Before we investigate how no arbitrage and the law of one price allow us to study the valuation, risk and risk management practices of fixed income instruments, let's take a closer look at the fixed income markets, using the entries in Table 1.2 as a guide. We begin with government debt, appearing under the heading of U.S. T-Bills and U.S. Bonds in Table 1.2.

1.2 THE GOVERNMENT DEBT MARKETS

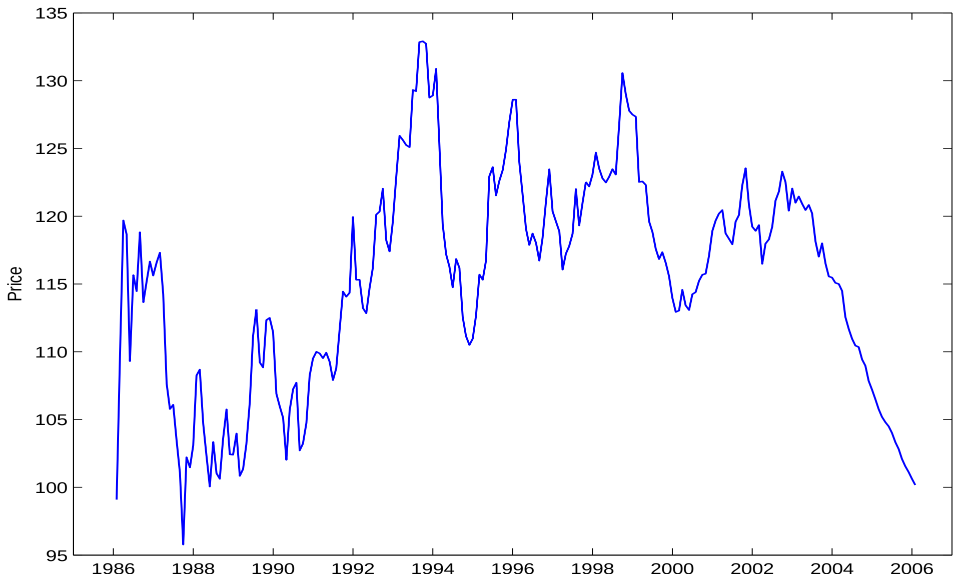
Essentially all countries issue debt to finance their operations. U.S. government debt has always occupied a special place in fixed income markets, mainly because it is perceived to have an extremely low probability of default. That is, investing in U.S. Treasury securities is considered "safe", as the government *will* repay its debt to investor. The quotation marks around the word "safe" underly an important caveat, though, which is what makes the analysis of fixed income securities so interesting. A U.S. Treasury bond can be considered a "safe" investment in terms of its risk of default: As noted, the issuer will in all likelihood repay its debt to its creditors (investors). The rationale behind its default safety is that these bonds are backed by the ability of the U.S. government to levy taxes on its citizens in the future to pay the debt back.

However, an investment in a U.S. Treasury bond may *not* be safe in terms of its return on investment over a short period of time. To provide an example, Figure 1.3 plots the life cycle of a 20-year bond, from its issuance in February 1986 to its maturity on February 2006.[1] The variation over time of the price of the bond is quite stunning, with run ups of over 30 percent within two years (e.g., between 1991 and 1993), and run downs at even higher speed (e.g., 1994). An investor buying this bond in 1993 would have suffered severe capital losses within the next year or so.

In addition to the potential capital losses in the bond price during a shorter period of time than the bond's maturity, an investment in U.S. Treasury securities entails additional sources of risk. The first is that most of the Treasury securities are nominal securities, that is, they pay coupons and principal in dollars. Therefore, if between the purchase of

[1]Data excerpted from CRSP (Daily Treasuries) ©2009 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business.

Figure 1.3 The February 2006, 9.375%, 20-Year Bond Price Path



Source: Center for Research in Security Prices

the bond and its maturity (e.g., 30 years) the U.S. enters a period of sustained inflation, the effective value of coupons and principal decreases, as investors cannot purchase as many consumption goods. This inflation risk must be taken into account in the analysis of Treasury securities. A related risk concerns the fact that the coupon and principal payments are in U.S. dollars, which may entail a currency risk for an overseas investor. For instance, European investors who purchased safe U.S. Treasury notes in 2005 have been hurt by the devaluation of the dollar with respect to the euro between 2006 and 2008.

The U.S. government issues various types of securities. Table 1.3 lists the types of securities. Treasury bills (T-bills) are short-term debt instruments, with maturity up to one year.[2] They do not pay any cash flow over time, only the principal at maturity. Treasury bills are issued very frequently, typically every week for bills up to six months, and every four weeks for one-year bills.

Treasury notes (T-notes) are medium-long term debt instruments, with maturity up to 10 years. These notes carry a fixed coupon that is paid semi-annually up to the maturity of the note. They are issued every month, except the ten-year note, that is issued less frequently. Treasury bonds (T-bonds) are longer-term debt instruments with maturity of 30 years at issuance. As with Treasury notes, bonds also carry a semi-annual coupon. The Treasury issues these long term bonds every six months.

In 1997 the U.S. government started issuing TIPS – Treasury Inflation Protected Securities – that is, securities that are indexed to inflation. Investors in T-notes and T-bonds are subject to inflation risk: Because the payment of coupons and final principal is in nominal

[2]The Treasury suspended the issuance of one year Treasury bills from August 2001 to June 2008.

Table 1.3 U.S. Treasury Debt Securities

Name	Maturity	Coupon Rate	Principal
Treasury Bills	4, 13, 26, and 52 weeks	None	fixed
Treasury Notes	2, 5, and 10 years	Fixed, semi-annual	Fixed
Treasury Bonds	30 years	Fixed, semi-annual	Fixed
TIPS	5, 10, and 20 years	Fixed, semi-annual	Adjusted for inflation

terms (i.e., simply U.S. dollars), if inflation increases substantially during the life of the debt instruments, these sums of money will be able to buy less of consumer goods. The TIPS offer protection to investors against this possibility: Because the principal is adjusted for inflation, higher inflation translates into both a higher final payoff at maturity of the bond, and higher coupons as well, since the coupon is defined as a fixed percentage of *current* principal (which increases with inflation). TIPS are issues with maturities of 5, 10 and 20 years.

The issuance calendar of the U.S. Treasury is very dense. Table 1.4 provides a snapshop of the issuing activity of the U.S. Treasury as of July 15, 2009. Market participants refer to the most recently issued Treasury securities as **on-the-run** securities, while all the others are called **off-the-run**. On-the-run Treasury securities tend to trade at a premium compared to similar off-the-run Treasury securities, which tend to be less liquid than the on-the-run securities.

1.2.1 Zero Coupon Bonds

Zero coupon bonds are securities that pay only the principal at maturity. A simple example is the Treasury bill described in Table 1.3. Other zero coupon bonds are available in the U.S. market through the STRIPS program. STRIPS (Separate Trading of Registered Interest and Principal Securities) are zero coupon bonds created from available U.S. Treasury notes and bonds by splitting the principal and each of the coupons from the bond. The U.S. Treasury does not issue these securities directly to investors, but investors can purchase them and hold them through financial institutions and government securities brokers and dealers. As an example of the available STRIPS on a particular date, Table 1.5 reports the stripped coupons available on September 25, 2008. The stripped coupon are the zero coupon bonds that are created only from the coupon interest payments of Treasury notes and bonds. In addition, a similar table is available for the stripped principals. The availability of these zero coupon bond securities with maturity up to 30 years enables investors to be more effective in their investment strategies and in their risk management practices, as we will discuss in later chapters.

1.2.2 Floating Rate Coupon Bonds

The bonds issued by the U.S. government have a fixed coupon rate. A floating rate coupon bond is like a standard coupon bond, but its coupon is indexed to some other short-term interest rate, which changes over time. While the U.S. government does not issue floating rate bonds, other governments do. For instance, Italy issues the CCT bond, which is an Italian Treasury debt security whose coupon rate is indexed to the six month rate of Italian 6-

Table 1.4 Issuance Acitivity of Bonds, Notes, and TIPS: January 15, 2009 to July 15, 2009

Security	Term	Type	Issue Date	Maturity Date	Interest Rate %	Yield %	Price per $100	CUSIP
3-YEAR		NOTE	7/15/2009	7/15/2012	1.5	1.519	99.944485	912828LB4
9-YEAR	10-MONTH	NOTE	7/15/2009	5/15/2019	3.125	3.365	97.998772	912828KQ2
10-YEAR		TIPS	7/15/2009	7/15/2019	1.875	1.92	99.592335	912828LA6
29-YEAR	10-MONTH	BOND	7/15/2009	5/15/2039	4.25	4.303	99.104142	912810QB7
2-YEAR		NOTE	6/30/2009	6/30/2011	1.125	1.151	99.94874	912828LF5
5-YEAR		NOTE	6/30/2009	6/30/2014	2.625	2.7	99.651404	912828KY5
7-YEAR		NOTE	6/30/2009	6/30/2016	3.25	3.329	99.510316	912828KZ2
3-YEAR		NOTE	6/15/2009	6/15/2012	1.875	1.96	99.753523	912828KX7
9-YEAR	11-MONTH	NOTE	6/15/2009	5/15/2019	3.125	3.99	92.968581	912828KQ2
29-YEAR	11-MONTH	BOND	6/15/2009	5/15/2039	4.25	4.72	92.50169	912810QB7
2-YEAR		NOTE	6/1/2009	5/31/2011	0.875	0.94	99.871675	912828KU3
5-YEAR		NOTE	6/1/2009	5/31/2014	2.25	2.31	99.718283	912828KV1
7-YEAR		NOTE	6/1/2009	5/31/2016	3.25	3.3	99.689717	912828KW9
3-YEAR		NOTE	5/15/2009	5/15/2012	1.375	1.473	99.713432	912828KP4
10-YEAR		NOTE	5/15/2009	5/15/2019	3.125	3.19	99.44721	912828KQ2
30-YEAR		BOND	5/15/2009	5/15/2039	4.25	4.288	99.36198	912810QB7
2-YEAR		NOTE	4/30/2009	4/30/2011	0.875	0.949	99.853739	912828KL3
5-YEAR		TIPS	4/30/2009	4/15/2014	1.25	1.278	100.113235	912828KM1
5-YEAR		NOTE	4/30/2009	4/30/2014	1.875	1.94	99.691687	912828KN9
7-YEAR		NOTE	4/30/2009	4/30/2016	2.625	2.63	99.968223	912828KR0
3-YEAR		NOTE	4/15/2009	4/15/2012	1.375	1.385	99.970714	912828KK5
9-YEAR	9-MONTH	TIPS	4/15/2009	1/15/2019	2.125	1.589	103.325496	912828JX9
9-YEAR	10-MONTH	NOTE	4/15/2009	2/15/2019	2.75	2.95	98.298568	912828KD1
2-YEAR		NOTE	3/31/2009	3/31/2011	0.875	0.949	99.853739	912828KH2
5-YEAR		NOTE	3/31/2009	3/31/2014	1.75	1.849	99.529266	912828KJ8
7-YEAR		NOTE	3/31/2009	3/31/2016	2.375	2.384	99.942292	912828KT6
3-YEAR		NOTE	3/16/2009	3/15/2012	1.375	1.489	99.667005	912828KG4
9-YEAR	11-MONTH	NOTE	3/16/2009	2/15/2019	2.75	3.043	97.504473	912828KD1
29-YEAR	11-MONTH	BOND	3/16/2009	2/15/2039	3.5	3.64	97.456658	912810QA9
2-YEAR		NOTE	3/2/2009	2/28/2011	0.875	0.961	99.830481	912828KE9
5-YEAR		NOTE	3/2/2009	2/28/2014	1.875	1.985	99.479306	912828KF6
7-YEAR		NOTE	3/2/2009	2/29/2016	2.625	2.748	99.22194	912828KS8
3-YEAR		NOTE	2/17/2009	2/15/2012	1.375	1.419	99.871395	912828KC3
10-YEAR		NOTE	2/17/2009	2/15/2019	2.75	2.818	99.411068	912828KD1
30-YEAR		BOND	2/17/2009	2/15/2039	3.5	3.54	99.264139	912810QA9
2-YEAR		NOTE	2/2/2009	1/31/2011	0.875	0.925	99.901394	912828JY7
5-YEAR		NOTE	2/2/2009	1/31/2014	1.75	1.82	99.667162	912828JZ4
20-YEAR		TIPS	1/30/2009	1/15/2029	2.5	2.5	99.063837	912810PZ5
3-YEAR		NOTE	1/15/2009	1/15/2012	1.125	1.2	99.77965	912828KB5
9-YEAR	10-MONTH	NOTE	1/15/2009	11/15/2018	3.75	2.419	111.579767	912828JR2

Source: U.S.Treasury Web Site http://www.treasurydirect.gov/RI/OFNtebnd accessed on July 16, 2009.

Table 1.5 Stripped Coupon Interest on September 25, 2008

Maturity							Maturity						
Year	Month	Day	Bid	Ask	Chg	Asked Yield	Year	Month	Day	Bid	Ask	Chg	Asked Yield
2008	11	15	99.898	99.918	0.001	0.6	2023	8	15	50.039	50.059	-0.487	4.7
2009	2	15	99.478	99.498	-0.068	1.31	2023	11	15	49.424	49.444	-0.489	4.71
2009	5	15	98.979	98.999	-0.056	1.59	2024	2	15	48.815	48.835	-0.529	4.71
2009	8	15	98.473	98.493	-0.146	1.72	2024	5	15	48.286	48.306	-0.532	4.71
2009	11	15	97.982	98.002	-0.194	1.78	2024	8	15	47.746	47.766	-0.553	4.7
2010	2	15	97.487	97.507	-0.236	1.83	2024	11	15	47.194	47.214	-0.555	4.7
2010	5	15	96.879	96.899	-0.277	1.93	2025	2	15	46.797	46.817	-0.408	4.69
2010	8	15	96.294	96.314	-0.318	2	2025	5	15	46.221	46.241	-0.371	4.69
2010	11	15	95.722	95.742	-0.359	2.05	2025	8	15	45.537	45.557	-0.372	4.71
2011	2	15	94.83	94.85	-0.413	2.23	2025	11	15	44.972	44.992	-0.297	4.72
2011	5	15	94.304	94.324	-0.442	2.23	2026	2	15	44.357	44.377	-0.297	4.73
2011	8	15	93.274	93.294	-0.539	2.42	2026	5	15	43.879	43.899	-0.298	4.72
2011	11	15	92.957	92.977	-0.481	2.34	2026	8	15	43.332	43.352	-0.298	4.73
2012	2	15	91.072	91.092	-0.48	2.78	2026	11	15	42.828	42.848	-0.299	4.73
2012	5	15	90.705	90.725	-0.515	2.69	2027	2	15	42.445	42.465	-0.262	4.71
2012	8	15	89.274	89.294	-0.566	2.94	2027	5	15	41.934	41.954	-0.263	4.72
2012	11	15	88.498	88.518	-0.589	2.97	2027	8	15	41.467	41.487	-0.263	4.71
2013	2	15	87.478	87.498	-0.607	3.07	2027	11	15	41.025	41.045	-0.206	4.71
2013	5	15	86.684	86.704	-0.647	3.1	2028	2	15	40.685	40.705	-0.11	4.69
2013	8	15	85.988	86.008	-0.666	3.11	2028	5	15	40.216	40.236	-0.11	4.69
2013	11	15	85.014	85.034	-0.725	3.18	2028	8	15	39.694	39.714	-0.111	4.7
2014	2	15	83.999	84.019	-0.763	3.26	2028	11	15	39.178	39.198	-0.111	4.71
2014	5	15	83.172	83.192	-0.814	3.29	2029	2	15	38.686	38.706	-0.092	4.71
2014	8	15	82.185	82.205	-0.828	3.36	2029	5	15	38.277	38.297	-0.033	4.71
2014	11	15	81.257	81.277	-0.903	3.41	2029	8	15	37.815	37.835	-0.033	4.71
2015	2	15	79.706	79.726	-0.462	3.58	2029	11	15	37.493	37.513	-0.015	4.69
2015	5	15	78.898	78.918	-0.489	3.6	2030	2	15	37.138	37.158	-0.015	4.68
2015	8	15	77.972	77.992	-0.502	3.64	2030	5	15	36.749	36.769	-0.015	4.68
2015	11	15	76.772	76.792	-0.525	3.73	2030	8	15	36.463	36.483	0.063	4.66
2016	2	15	75.885	75.905	-0.538	3.77	2030	11	15	36.084	36.104	0.063	4.66
2016	5	15	74.437	74.457	-0.573	3.9	2031	2	15	35.691	35.711	0.063	4.65
2016	8	15	73.593	73.613	-0.599	3.92	2031	5	15	35.282	35.302	-0.074	4.65
2016	11	15	72.086	72.106	-0.707	4.06	2031	8	15	34.957	34.977	-0.074	4.64
2017	2	15	71.16	71.18	-0.483	4.09	2031	11	15	34.656	34.676	-0.074	4.63
2017	5	15	70.144	70.164	-0.491	4.14	2032	2	15	34.438	34.458	-0.074	4.61
2017	8	15	69.036	69.056	-0.482	4.21	2032	5	15	34.264	34.284	-0.075	4.58
2017	11	15	68.213	68.233	-0.505	4.23	2032	8	15	33.918	33.938	-0.075	4.58
2018	2	15	67.643	67.663	-0.468	4.21	2032	11	15	33.556	33.576	-0.075	4.57
2018	5	15	66.816	66.836	-0.474	4.23	2033	2	15	33.337	33.357	-0.076	4.55
2018	8	15	65.674	65.694	-0.559	4.3	2033	5	15	32.984	33.004	-0.075	4.55
2018	11	15	64.851	64.871	-0.565	4.32	2033	8	15	32.674	32.694	-0.076	4.54
2019	2	15	63.626	63.646	-0.601	4.4	2033	11	15	32.329	32.349	-0.076	4.54
2019	5	15	62.904	62.924	-0.609	4.4	2034	2	15	32.008	32.028	-0.075	4.54
2019	8	15	61.826	61.846	-0.596	4.46	2034	5	15	31.651	31.671	-0.075	4.54
2019	11	15	61.081	61.101	-0.602	4.47	2034	8	15	31.357	31.377	-0.076	4.53
2020	2	15	60.194	60.214	-0.312	4.51	2034	11	15	31.008	31.028	-0.075	4.53
2020	5	15	59.29	59.31	-0.314	4.54	2035	2	15	30.662	30.682	-0.076	4.53
2020	8	15	58.475	58.495	-0.316	4.56	2035	5	15	30.321	30.341	-0.075	4.53
2020	11	15	57.716	57.736	-0.319	4.58	2035	8	15	29.983	30.003	-0.075	4.53
2021	2	15	56.876	56.896	-0.373	4.61	2035	11	15	29.649	29.669	-0.075	4.53
2021	5	15	56.128	56.148	-0.393	4.62	2036	2	15	29.339	29.359	-0.075	4.53
2021	8	15	55.368	55.388	-0.396	4.64	2036	5	15	29.268	29.288	0.161	4.49
2021	11	15	54.649	54.669	-0.416	4.65	2036	8	15	28.865	28.885	0.389	4.5
2022	2	15	53.989	54.009	-0.454	4.66	2036	11	15	28.664	28.684	0.161	4.49
2022	5	15	53.282	53.302	-0.475	4.67	2037	2	15	28.249	28.269	0.384	4.5
2022	8	15	52.599	52.619	-0.477	4.68	2037	5	15	28.212	28.232	0.161	4.47
2022	11	15	51.869	51.889	-0.479	4.7	2037	8	15	27.901	27.921	0.386	4.47
2023	2	15	51.144	51.164	-0.481	4.71	2038	2	15	27.331	27.351	0.23	4.46
2023	5	15	50.606	50.626	-0.484	4.7							

Source: *The Wall Street Journal*.

month T-bills (BOT). In addition, interest rate floaters are issued by financial institutions and corporations, as well as government agencies, such as the government mortgage companies Ginnie Mae, Freddie Mac, and Fannie Mae, within their collateralized mortgage obligations programs.

1.2.3 The Municipal Debt Market

The U.S. federal government issues debt to finance federal government expenses, such as health care and military expenses. Individual municipalities also issue debt independently to finance local projects. For instance, the City of Chicago issued bonds for $983,310,000 in 2003 to pay for an expansion project of its O'Hare International Airport.

The most interesting feature of "muni" bonds is that the interest income from their coupons are tax-exempt. As a consequence, the yield is lower than other regular Treasury notes and bonds, as the latter pay an income that is taxable according to investors' income tax rates.

1.3 THE MONEY MARKET

When we speak of the money market, we refer to the market for short-term borrowing and lending. Banks and financial institutions have various means of borrowing and lending at any point in time. The entries in Table 1.2 summarizes these channels.

1.3.1 Federal Funds Rate

Banks and other financial institutions must keep some amount of capital within the Federal Reserve. Balances at the Federal Reserve yield a small rate of return, which was in fact zero until September 2008. It is in the interest of banks to maintain their reserves as close to the limit as possible. Banks with a reserve surplus may then lend some of their reserves to banks with a reserve deficit. The effective Federal funds rate is the size-weighted average rate of interest that banks charge to each other to lend or borrow reserves at the Federal Reserve. Chapter 7 describes this market in more detail.

1.3.2 Eurodollar Rate

The Eurodollar rate is the rate of interest on a dollar deposit in a European-based bank. These are short-term deposits, ranging from 3 months to one year. In particular, the 90-day Eurodollar rate has become a standard reference to gauge the conditions of the interbank market. For instance, the market for Eurodollar futures and options, financial derivatives traded at the Chicago Mercantile Exchange that allow financial institutions to bet on or hedge against the future evolution of the Eurodollar rate (see Chapter 6), is among the largest and most liquid derivative markets in the world.

1.3.3 LIBOR

LIBOR stands for London Interbank Offer Rate. The British Bankers Association publishes daily the LIBOR rates. These rates correspond to the average interest rate that banks charge to each other for short-term uncollateralized borrowing in the London market. The rates available are very similar to the Eurodollar rates (see Table 1.2). LIBOR is however one of the most important benchmark rates, used often as the reference index in the large over-the-counter derivatives market. As explained in Chapter 5, interest rate swaps, the single largest derivatives market (see Table 1.1), use LIBOR rates as the reference rates to determine the size of cash flows implied by a contract.

1.4 THE REPO MARKET

The last entry in the top row of Table 1.2 reports the repo and reverse repo rates. The repo market plays an important role in the fixed income industry, as it is used by traders to borrow and lend cash on a collateralized basis. Because borrowing is collateralized, it is considered a safer way to lend cash, and this contributed to its growth over the years, making the repo market one of the most important sources of financing for traders. First of all, the formal definition:

Definition 1.2 *A **repurchase agreement (repo)** is an agreement to sell some securities to another party and buy them back at a fixed date and for a fixed amount. The price at which the security is bought back is greater than the selling price and the difference implies an interest rate called the* repo rate.

*A **reverse repo** is the opposite transaction, namely, it is the purchase of the security for cash with the agreement to sell it back to the original owner at a predetermined price, determined, once again, by the repo rate.*

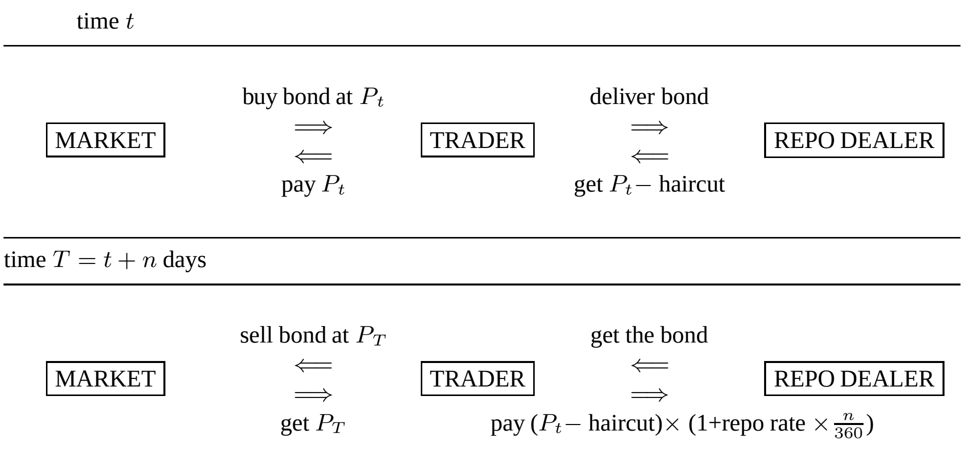
The best way to understand a repo transaction is to consider it as collateralized borrowing. A trader entering into a repo transaction with a repo dealer is borrowing cash (the sale price) in exchange for the security, which is held hostage by the repo dealer. If at the end of the repo term the trader were to default, the repo dealer could sell the security and be made whole. The following example illustrates the trade:

■ **EXAMPLE 1.1**

Suppose that a trader on September 18, 2007 (time t) wants to take a long position until a later time T on a given U.S. security, such as the 30-year Treasury bond. Let P_t denote the (invoice) price of the bond at time t. Figure 1.4 provides a schematic representation of the repo transaction: At time t, the trader buys the bond at market price P_t and enters a repurchase agreement with the repo dealer. Hence, the trader delivers the bond as collateral to the repo dealer and receives the cash to purchase the bond. In fact, the repo dealer typically gives something less than the market price of the bond, the difference being called a *haircut*. At time t the trader and the repo dealer agree that the trader will return back the amount borrowed, $(P_t - haircut)$, plus the repo rate.

What happens then at time T? At time T, the trader gets back the bond from the repo dealer, sells the bond in the market to get P_T and pays $(P_t - \text{haircut})$ plus the repo interest to the dealer. The repo interest is computed as the repo rate agreed at time t times the time between t and T. For instance, if n days pass between the two

Figure 1.4 Schematic Repo Transaction



dates, we have

$$\text{Repo interest} = \frac{n}{360} \times \text{Repo rate} \times (P_t - \text{haircut}) \tag{1.1}$$

where the denumerator "360" stems from the day count convention in the repo market.

The profit to the trader is then $P_T - P_t -$Repo interest. In percentage terms, the trader only put up the haircut (the margin) as own capital. Hence, the return on capital is

$$\text{Return on capital for trader} = \frac{P_T - P_t - \text{Repo interest}}{\text{Haircut}}$$

The position is highly leveraged and entails quite large risks. The case study in Section 3.7 of Chapter 3 discusses the risk and return of such leveraged transactions.

The term T of the repo transaction is decided at initiation, i.e., time t. In particular, most repurchase agreements are for a very short term, mainly overnight. However, as shown in Table 1.2, longer-term agreements reach 30 days or even more. Recall also that the repo rate is decided at time t.

Between t and T the trader (who is long the bond) earns the interest that accrues on the bond. Because the trader has to pay the repo rate during this period, setting up the repo transaction tends to generate a positive or negative stream of payments, depending on whether the interest earned on the bond is above or below the repo interest. We say that the trade implies a *positive carry* if the interest on the bond is above the repo rate and *negative carry* if the interest on the bond is below the repo rate.

1.4.1 General Collateral Rate and Special Repos

Other important definitions and characteristics of repo markets are as follows:

1. **General Collateral Rate (GCR)**: This is the repo rate on most Treasury securities, such as the off-the-run Treasuries. Because most Treasury securities have similar

characteristics in terms of liquidity, market participants require the same interest rate for collateralized borrowing.

2. **Special Repo Rate:** At times, one particular Treasury security is in high demand and hence the repo rate on that security falls to a level substantially below the GCR. As an example, on-the-run Treasury securities typically are "on special," in the sense that the repo rate charged for collateralized borrowing is smaller than the GCR.

Why does a security that is in high demand entail a lower (special) repo rate? To understand the logic, consider next example, which entails a reverse repo, whose rates are also quoted in Table 1.2.

■ EXAMPLE 1.2

Consider a trader who thinks a particular bond, such as the on-the-run 30-year Treasury bond, is overpriced and wants to take a bet that its price will decline in the future. If the trader does not have the bond to sell outright, then he or she can enter into a reverse repo with a repo dealer to obtain the bond to sell. More specifically, in a reverse repo, the trader essentially (A) borrows the security from the dealer; (B) sell it in the market; and (C) post cash collateral with the dealer. Figure 1.5 shows a schematic representation of the trade.

The trader is now *lending* money to the repo dealer against the bond. Therefore, the trader is now entitled to receive the repo rate. However, the trader, who wants to speculate on the decrease in the bond price, is happy to forgo part or all of the repo rate in order to get hold of the bond. If many traders want to undertake the same strategy of shorting that particular bond, then that bond is in high demand, and the repo rate for *that* bond declines below the general collateral rate. That bond is said to be "on speacial." The profit from the reverse repo transaction is then

$$\text{Profit} = (P_t - P_T) + \text{Repo interest}$$

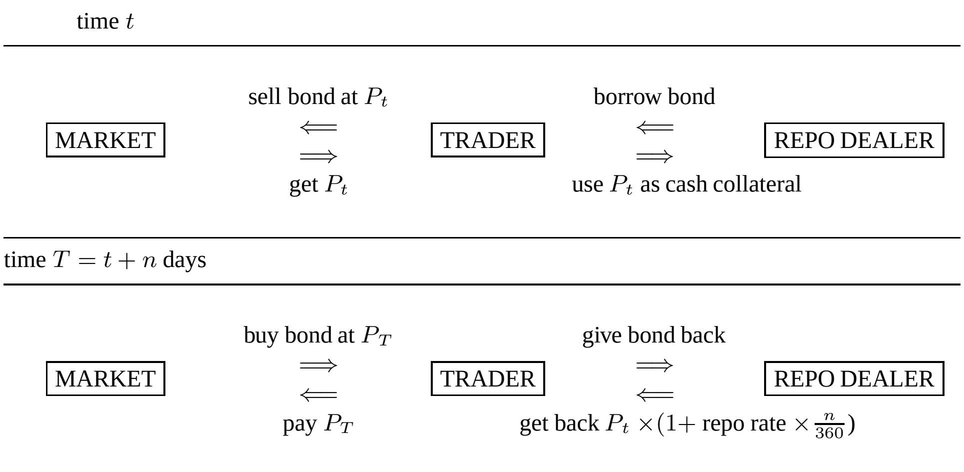
where the repo interest is computed as in Equation 1.1, namely, the amount deposited with the repo dealer (P_t) times the repo rate times $n/360$, where n is the number of days between the two trading dates t and T.

As mentioned, the repo market has grown steadily over the years. Table 1.6 shows the average *daily* amount outstanding in these contracts.[3] Since borrowing is collateralized by the value of the asset, the repo rate is lower than other borrowing rates available to banks, such as LIBOR. Figure 1.6 plots the time series of the one month and the three month T-Bill, Repo and LIBOR rates from May 1991 to April 2008. As it can be seen, for both maturities, the safe T-bill rate is the smallest and the LIBOR is the highest of the three rates, as borrowing and lending at the LIBOR rate is riskier as the loans are uncollateralized.

We cover additional details regarding the repo market and its uses by market participants in a number of case studies. For instance, in Chapters 3 and 4 we discuss the use of repurchase agreements to increase portfolio leverage, in Chapter 5 we illustrate the use of the repo market to carry out a swap spread arbitrage trade, and in Chapter 16 we use

[3] The amount outstanding of repurchase agreements need not equal the amount outstanding of reverse repurchase agreements, as each column reflects the size of collateralized borrowing or lending of security dealers only, and not the whole universe of repo counterparties.

Figure 1.5 Reverse Repo Transaction



repo transactions to implement a relative value arbitrage trade on the yield curve through a dynamic long/short strategy.

1.4.2 What if the T-bond Is Not Delivered?

Consider the reverse repurchase agreement displayed in Figure 1.5. At maturity of the repo contract (time T), the trader must return the bond to the repo dealer in exchange for the cash amount $P_t \times (1 + \frac{n}{360} \times \text{repo rate})$. What happens if the trader does not return the security? Such an occurrence is called a **fail**, and up to May, 2009 such a failure to deliver would have simply implied that the repo dealer in this example would have kept the cash received, P_t, plus the repo interest. The cost for failing to deliver for the trader was simply to forgo the repo interest. The financial crisis of 2007 - 2009 led the Federal Reserve to lower the reference Fed funds rate to close to zero, and repo rates also fell to essentially zero. When the repo rate is zero, however, the cost for a trader to fail to deliver the bond is very small, as the trader may keep the bond itself if this bond is particularly valuable. The financial crisis of 2007 - 2009 generated a 'flight-to-quality,' meaning that investors dumped all risky securities and strongly demanded safe U.S. Treasuries, as the demand for safe collateral increased. This increase in demand for U.S. Treasuries made it difficult or costly for traders who have short positions to find the bonds to return to their counterparties in the reverse repo transactions. Given the small cost of failing to deliver, the number of fails spiked in the last quarter of 2008. Figure 1.7 reports the cumulative weekly failures of delivering Treasury securities, in millions of dollars, by primary dealers, and the 2008 spike is clearly visible. The figure however also shows that sustained periods of delivery fails occurred in the past as well, such as in 2001 and in 2003. We should mention that a spike in delivery fails can also be due to a snowball effect, as the failure to deliver from a security dealer implies that another security dealer who was counting on the delivery to

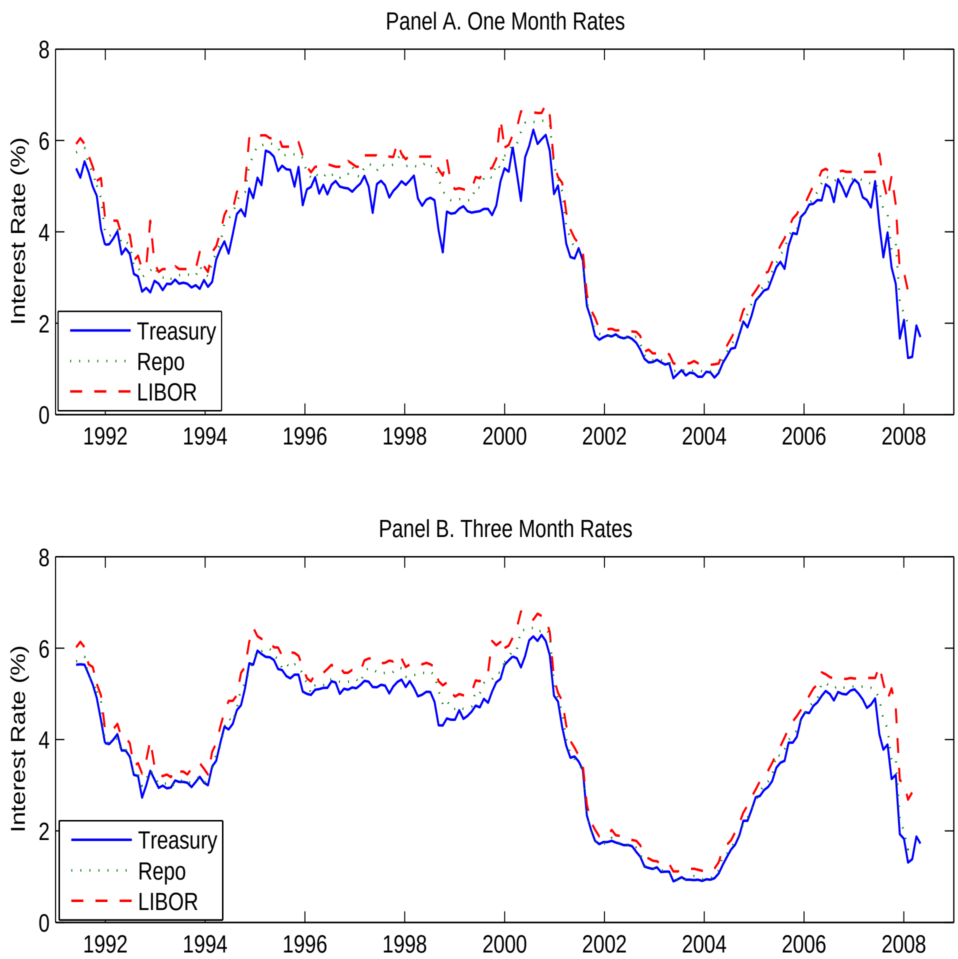
Table 1.6 Financing by U.S. Government Securities Dealers

Reverse Repurchase and Repurchase Agreements (1) Average Daily Amount Outstanding 1981 - 2006 ($ Billions)			
	Reverse Repurchase	Repurchase	Total
1981	46.7	65.4	112.1
1982	75.1	95.2	170.3
1983	81.7	102.4	184.1
1984	112.4	132.6	245
1985	147.9	172.9	320.8
1986	207.7	244.5	452.2
1987	275	292	567
1988	313.6	309.7	623.3
1989	383.2	398.2	781.4
1990	377.1	413.5	790.5
1991	417	496.6	913.6
1992	511.1	628.2	1,139.3
1993	594.1	765.6	1,359.7
1994	651.2	825.9	1,477.1
1995	618.8	821.5	1,440.3
1996	718.1	973.7	1,691.8
1997	883	1,159.0	2,042.0
1998	1,111.4	1,414.0	2,525.5
1999	1,070.1	1,361.0	2,431.1
2000	1,093.3	1,439.6	2,532.9
2001	1,311.3	1,786.5	3,097.7
2002	1,615.7	2,172.4	3,788.1
2003	1,685.4	2,355.7	4,041.1
2004	2,078.5	2,868.2	4,946.7
2005	2,355.2	3,288.4	5,643.6
2006	2,225.2	3,388.3	5,613.5

(1) Figures cover financing involving U.S. government, federal agency, and federal agency MBS securities.

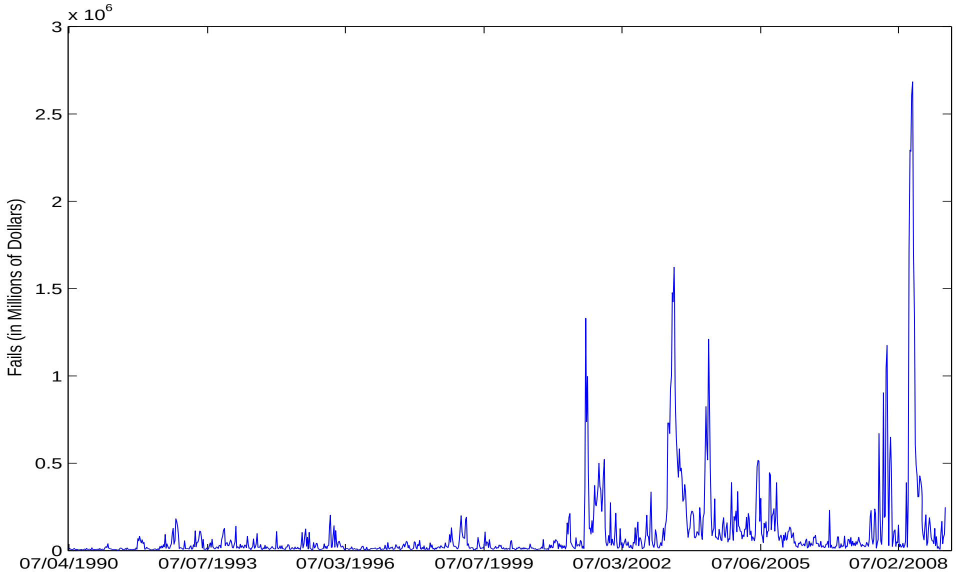
Source: Federal Reserve Bank of New York
obtained from SIFMA web site: *http://www.sifma.net/story.asp?id=1176*

Figure 1.6 Short-Term Rates: 1991 - 2008



Source: Federal Reserve Board, British Bankers Association, Bloomberg.

Figure 1.7 Primary Dealers Fails to Deliver: 1990 to 2009



Source: Federal Reserve Bank of New York.

settle his or her own obligation may be unable to deliver as well, and so on, generating a domino effect.[4]

Starting May 1, 2009, the Federal Reserve imposed a penalty charge of 3% over its Fed funds rate for failing to deliver the bonds in the repo transactions the Federal Reserve conducts daily in its open market operations (see Chapter 7 for details on the Fed conduct of monetary policy). In addition, the Federal Reserve has been encouraging market participants to adopt a similar charge as part of best practices in repo market transactions. An interesting outcome of such penalty on a fail to deliver is that on May 1, 2009, the repo rate for some Treasury securities that were on special became negative.[5] How is it possible that an interest rate is negative? The reason is that a trader that has to deliver a given Treasury security to a conterparty is willing to pay to get hold of the security rather than incurring the penalty. In particular, the trader can enter into a reverse repo with another repo dealer at a negative repo rate to obtain the Treasury security to deliver to the original counterparty.

1.5 THE MORTGAGE BACKED SECURITIES MARKET AND ASSET-BACKED SECURITIES MARKET

One of the interesting patterns evident in Figure 1.1 is the dramatic growth experienced by the mortgage backed securities market, which hit the $8.9 trillion mark by the end of 2008. Chapter 8 describes this market in detail as well as the type of securities that are exchanged

[4]See the "Guide to FR2004 Settlement Fails Data," Federal Reserve of New York. See also the article by Michael J. Fleming and Kenneth D. Garbade, "When the Back Office Moved to the Front Burner: Settlement Fails in the Treasury Market after 9/11," Federal Reserve Bank of New York Economic Policy Review, November 2002.
[5]See Bloomberg.com Web site http://www.bloomberg.com/apps/news?pid=20601009&sid=a85sg4IKcjCM.

in it. However, as a brief introduction, the source of the mortgage-backed securities market is relatively simple: homeowners across the U.S. finance their homes through mortgages, issued by local savings & loans, thrifts, and other banks. When a bank issues a mortgage to a homeowner, the mortgage rests on the asset side of the bank's balance sheet. The mortgage is a fixed income instrument: It is a promise from the homeowner to make certain cash payments in the future. These cash payments are affected by numerous events, discussed further in Chapter 8, which make them risky for the bank. In particular, if a local bank provides mortgages to a local community only, it is subject to the risk that these homeowners may all default at the same time because of local geographical factors. For instance, if the local community is highly specialized in a particular industrial sector, and the latter goes into an economic crisis, one could expect large layoffs in that community, which in turn would increase the probability that homeowners will default on their mortgage payment obligations. Similarly, if the house prices of that particular community decline, the collateral in the mortgage contract declines, and the local bank is then in a more risky position than before.

Mortgage backed securities allow a bank to diversify this risk. The idea is to resell its mortgages, now on the asset side of the bank, for cash. In order to improve the liquidity and to mitigate credit risk, the market evolved into one in which *many* similar mortgages are pooled together to form a large collateral of assets. These assets, which have better characteristics in terms of diversification of risk, make up the collateral on debt securities issued to individual investors, called mortgage backed securities. In summary, an investor in a mortgage backed security obtains a legal claim to the cash flows (coupons) that are paid by the original homeowner.

The mortgage backed securities trade in the market. For instance, in Table 1.2, the heading "30 Y MBS" reports the prices of popular mortgage-backed securities, those issued by Ginnie Mae (GNMN 6.0), Freddie Mac (GOLD 6.0) and Fannie Mae (FNMA 6), the three largest players in the mortgage-backed securities market. From an investment perspective, a large part of the mortgage-backed securities market is considered default free, because these three big players – Ginnie, Freddie and Fannie – have an implicit or explicit backing of the full faith of the U.S. government. Indeed, while Ginnie Mae has always been a government entity, Fannie and Freddie entered conservatorship in September 2008, which implies that their own debt securities but especially their mortgage-backed securities are default free. Still, compared to Treasury debt securities mortgage-backed securities have many peculiarities regarding the timing of promised cash flows, which may vary unexpectedly due changes in interest rates, or changes in housing prices, or a severe recession. These unexpected variations in cash flows make mortgage-backed securities risky and, for this reasons, such securities typically offer an additional return on investment, compared to Treasury securities. Chapter 8 discusses this market in more detail.

Similarly to the mortgage backed securities market, the asset backed securities market involves the issuance of debt instruments to investors, collateralized by various types of loans, such as auto loans, credit cards, and the like. The market is smaller in size, as shown in Table 1.1. This market is considered by market participants riskier, as they do not involve

Table 1.7 Borrowing rates for firm A and B

	Firm A	Firm B
Fixed Rate	15%	12%
Floating Rate	LIBOR + 3%	LIBOR + 2%

1.6 THE DERIVATIVES MARKET

Table 1.2 also reports quotes of several derivative securities. As show in Table 1.1, the interest rate derivatives market is huge and it has been growing steadily for the past three decades, as illustrated in Figure 1.2. Chapters 5 and 6 explore these markets in more detail, and provide some early examples of uses of derivative contracts for corporations and traders. Additional examples discussing the fair valuation and the risk of derivative securities are offered throughout the book, as we explore the modeling devices applied by financial institutions to price and hedge these securities.

The swap contract is the largest market of all. While future chapters discuss the pricing, hedging and the risk involved in swaps, it is informative at this point to see the economic need that led to the creation of this market at the beginning of the 1980s.

1.6.1 Swaps

Interest rate swap contracts were introduced in the early 1980s to take advantage of some apparent arbitrage opportunity that was surfacing in the corporate bond market. The following is a stylized example of common situations occurring at that time.

EXAMPLE 1.3

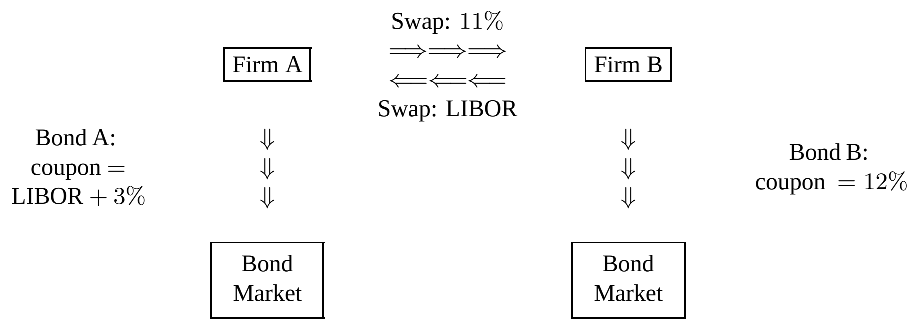
Consider the following situation. There are two firms, Firm A and Firm B. Firm A wants to raise $M =$\$10 million using fixed rate coupon bonds, while Firm B wants to raise $M =$\$10 million using floating rate coupon bonds. Let the market rates available to the two firms be the ones in Table 1.7. That is, Firm A can either borrow at a fixed coupon rate of 15% or at a floating rate with coupon linked to the six-month LIBOR rate plus 3%. Firm B, instead, can borrow fixed at 12% or floating at LIBOR + 2%. Note that the rates available to B are always lower than the ones available to A, reflecting a difference in credit risk.

An investment bank observing the rates in Table 1.7 may offer the following deal to the two firms.

First, Firm A issues a floating rate bond at LIBOR + 3%, while Firm B issues a fixed rate bond at 12%. Second, the two firms swap coupon payments. In particular, they could consider the following swap deal:

- Firm A pays B a fixed rate payment at 11% per year; and
- Firm B pays A a floating rate payment at LIBOR.

Figure 1.8 A Swap Deal



Consider now the net cash flow from each of these firms when we take together the bond issuance and the swap deal. For every future coupon date, we have:

$$\begin{aligned} \text{Firm A Pays} &: \underbrace{(\text{LIBOR } + 3\%)}_{\text{to market}} + \underbrace{11\% - \text{LIBOR}}_{\text{swap deal}} = 14\% \\ \text{Firm B Pays} &: \overbrace{\qquad 12\% \qquad} \quad \overbrace{-11\% + \text{LIBOR}} = \text{LIBOR} + 1\% \end{aligned}$$

Figure 1.8 shows the cash flows in every period. Overall, we observe that Firm A pays 14% instead of 15%, which was its fixed coupon market rate. Similarly, Firm B pays LIBOR $+1\%$ instead of LIBOR $+2\%$, which was its market rate. Both firms then gain from entering this deal.

The early swaps managed to arbitrage some relative price discrepancy that existed between floating rates and fixed rates. The notion was that if a spread exists because of the risk of default, it should be the same across floating and fixed coupon bonds. The difference in spreads between the two firms across asset classes, floating and fixed coupon bonds, generates the possibility of a trade. How can we compute the gains from a swap trade? The size of the pie that the two firms can divide through a swap is given by the difference in comparative advantage implicit in Table 1.7. In other words, the spread on the fixed coupon is 3%($= 15\% - 12\%$) while the spread on floating coupon is only 1%($=$ (LIBOR $+3\%$) $-$ (LIBOR $+1\%$)). The difference in spreads provide the total gains that can be split between the two firms:

$$\text{Gains from trade} = \text{Fixed spread} - \text{Floating spread} = 3\% - 1\% = 2\% \tag{1.2}$$

In the example we divided the gains from trade equally between the two firms. In reality, the exact split depends on the relative contractual strength of each firm: Firms with higher creditworthiness would tend to get a higher coupon.

In addition, some part of the gain would also accrue to the investment bank that brokers the deal.

The initial spur to swap trading was due to exploit arbitrage opportunities. At that time, investment banks would also reap substantial profits from relatively large spreads. However, as we will discuss in more detail in Chapter 5 and elsewhere, the growth in the swap market came about because of the extreme usefulness of swaps as a convenient means for cash management and risk management. Financial institutions, corporations, and even governments use swaps (a) to change the sensitivity of their cash flows to fluctuations in interest rates; (b) to alter the timing of their payments and revenues; (c) or even simply for investment purposes within complex trading strategies.

1.6.2 Futures and Forwards

Table 1.2 also reports the quotes of futures contracts, for instance, the "90 Day Eurodollar Futures,"Eurodollar,futures the "Fed Funds Futures," and the "10-year Treasury Note Futures." Futures contracts, discussed in detail in Chapter 6, are contracts according to which two counterparties decide to exchange a security, or cash, or a commodity, at a prespecified time in the future for a price agreed upon today. The quote represents the price at which delivery will take place in the future. For interest rate futures, such as Eurodollar or Fed funds, the quoted "price" is given by "$100-$ futures rate." For instance, the 90-day Eurodollar futuresEurodollar,futures contract with maturity "December," quoted at 95.07 in Table 1.2, establishes the rate today, 4.93% $= 100 - 95.07$, at which the party long the futures could deposit dollars in the Eurodollar market in December for the following 90 days.[6] The futures market thus provides a convenient way for market participants to lock-in a future interest rate: For instance, a corporation that has a large receivable due in December can exploit the futures market to lock in the rate (4.93%) at which it can park the sum of money for the following 90 days. In addition, the futures market is often used by market participants to gauge the market expectation about future movement in interest rates. Important in futures contracts is the fact that either counterparty may be called to make payments in the future.

Table 1.2 does not report quotes of forward contracts, although we see from Table 1.1 that forward contracts make up a sizable share of the fixed income market. Forward contracts are similar to futures contracts, in that two counterparties agree today that they will exchange a security (or cash) in the future at a price that is decided today. Just like futures contracts, forward contracts allow institutions to lock in interest rates for the future. Unlike futures, forward contracts are not traded on regulated exchanges but only on the over-the-counter market. Chapters 5 and 6 describe these contracts and delve into the differences between futures and forwards.

1.6.3 Options

Table 1.2 does not provide any quotes for interest rate options, as we obtained this table from the BTTM screen from a Bloomberg terminal, which only reports U.S. Treasury and money market rates. But options contracts are a vital part of the fixed income market. Table 1.1 shows that indeed the options' market is quite larger than the futures and forward markets. In addition, options are implicitly embedded in several other securities, such as callable bonds, mortgage backed securities, and other types of structured notes. But first of all:

What is an option?

[6]In fact, this futures is cash settled, so the deposit does not actually need to take place. See Chapter 6.

Intuitively, an option is the financial equivalent of an insurance contract: It is a contract according to which the option buyer, who purchases the insurance, receives a payment from the option's seller, who sold the insurance, only if some interest rate scenario occurs in the future. For instance, a corporation that issues a floating rate bond – a bond whose coupon is tied to the level of a short-term floating rate – may be worried about an interest rate hike in the future, a scenario that may drain too much financial resources from the coporation. The corporation may purchase insurance against such scenarios, by purchasing a financial option, called a cap, that pays only if the floating reference interest rate increases above some cutoff point, called the strike rate. This contract would be a good hedge for the firm against interest rate hikes, because, if the interest rate does increase above the strike rate, then the option's seller must pay the corporation a contractually agreed-upon cash flow, which the corporation can use to pay its own liability to its bond holders.

Many options are implicit in many securities. A homeowner who financed the purchase of his or her home using a adjustable rate mortgage (ARM), for instance, most likely also bought (probably unknowingly) an option against an increase in interest rates. The reason is that standard adjustable rate mortgages contain a provision stating that the maximum rate the homeowner will have to pay over the life of the mortgage is capped at some level. Therefore, the loan contract is equivalent to a standard floating rate loan contract plus an option that pays if interest rates become too high, just like in the example of the corporation above. Similarly, a homeowner who financed the purchase of his or her home using a fixed rate mortgage also bought an option to pay back the mortgage whenever he or she likes. In particular, homeowners pay back loans when the interest rate declines. The bank making the mortgage implicitly sold the option to the homeowner, and the option premium is embedded in the mortgage rate. Considering that the mortgage backed securities market has become the dominant fixed income market in the U.S. (its value as of December 2008 is about $9 trillion, compared to only $6 trillion of the U.S. debt), the understanding of the impact of options on fixed income instruments has never being more important.

1.7 ROADMAP OF FUTURE CHAPTERS

In this chapter we described some of the major fixed income markets. Starting with the next chapter, we begin to analyze each market in much more detail. In Chapter 2 we cover the basics of fixed income instruments, that is, the notion of a discount, of an interest rate, and how we compute the fair valuation of Treasury bills, notes, and bonds. At the end of the chapter we also show how we can use this information to obtain the price of some simple structured securities, such as inverse floaters, which are popular securities if an investor wishes to bet on a decrease in interest rates. Fixed income securities present many risks for investors, even if they are issued or guaranteed by the U.S. government and therefore they are default free. Indeed, long-term bonds, for instance, may suffer strong capital losses in response to a generalized increase in interest rates. Chapters 3 and 4 discuss the types of risk embedded in fixed income securities, the issue of risk measurement, as well as the practice of risk management, such as asset-liability management and immunization strategies. Chapters 5 and 6 cover popular fixed income derivatives, such as forward rates, swaps, futures, and options, and their uses by market participants. Chapter 7 links the fixed income market to the real economy. In particular, we talk about monetary policy, economic growth, and inflation. In this context, we also discuss the market for TIPS, the inflation-protected debt securities. Finally, Chapter 8 discusses the residential mortgage backed securities market, in terms of the types of securities as well as their riskiness from an investment perspective. This chapter concludes the first part of the book, which aims at providing some basic notions of fixed income securities.

Chapter 9 begins the second part of the book, which concerns the fair valuation of derivative securities by no arbitrage. In particular, we begin with simple, one-period binomial trees to explain the relations that have to exist between any pair of fixed income securities. In this chapter we also introduce a popular pricing methodology called risk neutral pricing. Chapter 10 expands the concepts of one-period binomial trees to multiple periods, and discusses the issue of dinamic hedging, the standard methodology of hedging a risk exposure by rebalancing the portfolio over time as the interest rate changes. Chapter 11 applies the methodology illustrated in the two earlier chapters to real-world securities. In particular, it covers some popular models for the pricing of fixed income instruments, as well as their estimation using real data. These concepts are further developed in Chapter 12, which details the pricing and hedging of an important class of derivatives, called American options. Such options are implicitly embedded in numerous debt securities, from callable bonds to mortgage backed securities. Chapter 13 illustrates a powerful methodology for valuing and hedging complicated securities, namely, the Monte Carlo simulations methodology. This methodology involves using computers to simulate interest rate paths and price paths and then using those simulated quantities to compute current prices and hedge ratios. We apply this methodology to real world securities, such as corridor notes, ammortizing index swaps, mortgage backed securities, and collateralized debt obbligations. This chapter concludes the second part of the book.

The third part of the book is mathematically more advanced, and some familiarity with advanced calculus is required. In particular, Chapter 14 introduces continuous time methodologies, the notion of a Brownian motion, and Ito's lemma. We apply these continuous time methodologies and the rules of no arbitrage in Chapter 15 to compute the fair valuation of Treasury notes and bonds, as well as derivative securities, such as options. Compared to the second part of the book, which also accomplishes similar goals, the concepts discussed in this part of the text are more realistic, and moreover provide analytical formulas for the pricing and hedging of numerous securities, a very convenient property for traders who are pressed for time. Chapter 16 discusses the notion of dynamic hedging, that is, the practice by market participants of frequently rebalancing their portfolios to hedge their risk exposure. Chapter 17 introduces the notion of risk neutral pricing in continuous time models. In addition, we develop for this class of models the Monte Carlo simulation approach to pricing and hedging securities, a methodology widely used by market participants. In well-functioning markets, any risk embedded in fixed income securities should be compensated for by a risk premium on its rate of return, either through a high coupon or a low purchasing price. The link between risk and return of fixed income securities is the object of Chapter 18. In particular, we discuss the fact that if a security is providing an above-market coupon or rate of return, then most likely this security is exposed to some risk, which perhaps is not made completely explicit. To make this point clearer, we discuss a famous case involving a special swap between Procter & Gamble and the investment bank Bankers Trust. Finally, Chapters 19 to 22 cover more advanced models for fixed income security pricing. These chapters discuss several examples in which models are applied to real world securities, and draw some distinctions among them.

1.8 SUMMARY

In this chapter we covered the following topics:

1. Arbitrage strategies: These are strategies that cost nothing to enter into, and provide sure money within a certain time. In well-functioning markets we should not expect arbitrage strategies to persist for a long time. Indeed, pure arbitrage strategies are rare in the market. The rules of no arbitrage determine the relative pricing across fixed income securities and explain their high correlation.

2. U.S. Treasury market: The U.S. issues four types of securities: short-maturity T-bills, medium-maturity T-notes, long-maturity T-bonds, and TIPS, the inflation-protected securities. The size of the U.S. debt market is no longer dominant in fixed income markets, as other markets became even larger, notably the mortgage backed securities market and the derivatives market.

3. Money markets and money markets rates: This market is the source of short-term borrowing by financial and non financial institutions. The main money market rates are the commercial paper rate, LIBOR, and the Federal funds rate. The LIBOR rate, the rate at which banks in London borrow from each other on an uncollateralized basis, is the main reference rate in numerous derivative securities.

4. Repurchase agreements and the repo rate: In collateralized borrowing between two counterparties, the repo rate is the borrowing or lending rate within a repurcahse agreement. Because the borrowing is collateralized, the rate is lower than the LIBOR rate, for instance.

5. Mortgage backed securities market: This is the largest debt market in United States. Mortgage backed securities are collateralized by pools of residential and non residential mortgages and sold to investors who then receive claims to the mortgages coupons. These securities present numerous additional risks for investors compared to Treasury securities.

6. Swaps market: A swap is a contract according to which two counterparties agree to exchange cash flows in the future. This market is very large in size, and although considered a derivative market, its sheer size makes it equivalent to a primary market, in the sense that the prices of swaps are really not derived from those of other securities, but rather they depend on the relative size of demand and supply of these contracts by market participants.

CHAPTER 2

BASICS OF FIXED INCOME SECURITIES

2.1 DISCOUNT FACTORS

Receiving a dollar today is not the same as receiving it in a month or in a year. There are numerous reasons why people would like to have money today rather than in the future. For one, money today can be put in a safe place (a bank or under the mattress) until tomorrow, while the opposite is not easily doable. That is, money in hands gives its holder the option to use it however he or she desires, including transferring it to the future through a deposit or an investment. This option has a value on its own. If we agree that households and investors value \$1 today more than \$1 in the future the question is then how much \$1 in the future is worth in today's money. The value of what \$1 in the future would be in today's money is called the discount factor. The notion of discount factors is at the heart of fixed income securities.

It is easiest to introduce the concept by looking at a concrete example. The U.S. government, as with most governments, needs to borrow money from investors to finance its expenses. As discussed in Chapter 1, the government issues a number of securities, such as Treasury bills, notes and bonds, to investors, receiving money today in exchange for money in the future. The U.S. Treasury is extremely unlikely to default on its obligations, and thus the relation between purchase price and payoff of U.S. Treasury securities reveals the market time value of money, that is, the exchange rate between money today and money in the future. Example 2.1 illustrates this point.

Fixed Income Securities. By Pietro Veronesi

EXAMPLE 2.1

On August 10, 2006 the Treasury issued 182-day Treasury bills. The issuance market price was \$97.477 for \$100 of face value.[1] That is, on August 10, 2006, investors were willing to buy for \$97.477 a government security that would pay \$100 on February 8, 2007. This Treasury bill would not make any other payment between the two dates. Thus, the ratio between purchase price and the payoff, $0.97477 = \$97.477/\$100$, can be considered the market-wide discount factor between the two dates August 10, 2006 and February 8, 2007. That is, market participants were willing to exchange 0.97477 dollars on the first date for 1 dollar six months later.

Definition 2.1 *The* **discount factor** *between two dates, t and T, provides the term of exchange between a given amount of money at t versus a (certain) amount of money at a later date T. We denote the discount factor between these two dates by $Z(t, T)$.*

In the above example, the two dates are $t =$ August 10, 2006 and $T =$ February 08, 2007. The discount factor is $Z(t, T) = 0.97477$.

In short, the discount factor $Z(t, T)$ records the time value of money between t and T. Since it is a value (what is the value today of \$1 in the future), it is essentially a price, describing how much money somebody is willing to pay today in order to have \$1 in the future. In this sense, the notion of a discount factor is un-ambiguous. In contrast, as we shall see below, the related notion of an interest rate is not un-ambiguous, as it depends on compounding frequency, for instance. Exactly because discount factors unambiguously represent a price – an exchange rate between money today versus money tomorrow – they are at the heart of fixed income securities analysis. In the following sections we describe their characteristics in more detail.

2.1.1 Discount Factors across Maturities

Definition 2.1 and Example 2.1 highlight that the discount factor at some date t (e.g., August 10, 2006) depends on its maturity T (e.g., February 8, 2007). If we vary the maturity T, making it longer or shorter, the discount factor varies as well. In fact, for the same reason that investors value \$1 today more than \$1 in six months, they also value \$1 in three months more than \$1 in six months. This can be seen, once again, from the prices of U.S. Treasury securities.

EXAMPLE 2.2

On August 10, 2006 the U.S. government also issued 91-day bills with a maturity date of November 9, 2006. The price was \$98.739 for \$100 of face value. Thus, denoting again $t =$ August 10, 2006, now $T_1 =$ November 9, 2006, and $T_2 =$ February 8, 2007, we find that the discount factor $Z(t, T_1) = 0.98739$, which is higher than $Z(t, T_2) = 0.97477$.

[1] These data are obtained from the Web site http://www.treasurydirect.gov/RI/OFBills, accessed on August 22, 2006.

This example highlights an important property of discount factors. Because it is always the case that market participants prefer $1 today to $1 in the future, the following is true:

Fact 2.1 *At any given time t, the discount factor is lower, the longer the maturity T. That is, given two dates T_1 and T_2 with $T_1 < T_2$, it is always the case that*

$$Z(t, T_1) \geq Z(t, T_2) \tag{2.1}$$

The opposite relation $Z(t, T_1) < Z(t, T_2)$ would in fact imply a somewhat curious behavior on the part of investors. For instance, in the example above in which $T_1 =$ November 9, 2006 and $T_2 =$ February 8, 2007, if $Z(t, T_1)$ was lower than $Z(t, T_2) = 0.97477$, it would imply that investors would be willing to give up $\$97.477$ today in order to receive $\$100$ in six months, but *not* in order to receive the same amount three months earlier. In other words, it implies that investors prefer to have $100 dollar in six months rather than in three months, violating the principle that agents prefer to have a sum of money earlier rather than later. Moreover, a violation of Relation 2.1 also generates an arbitrage opportunity, which we would not expect to last for long in well functioning financial markets (see Exercise 1). In Chapter 5 we elaborate on this topic, showing also that a violation of Relation 2.1 amounts to the assumption that future nominal interest rates be negative.

2.1.2 Discount Factors over Time

A second important characteristic of discount factors is that they are not constant over time, even while keeping constant the time-to-maturity $T - t$, that is, the interval of time between the two dates t and T in the discount factor $Z(t, T)$. As time goes by, the time value of money changes. For instance, the U.S. Treasury issued a 182-day bill on $t_1 =$ August 26, 2004, with maturity $T_2 =$ February 24, 2005, for a price of $99.115. This price implies a discount factor on that date equal to $Z(t_1, T_1) = 0.99115$. This value is much higher than the discount factor with the same time to maturity (six months) two years later, on August 10, 2006, which we found equal to 0.97477.

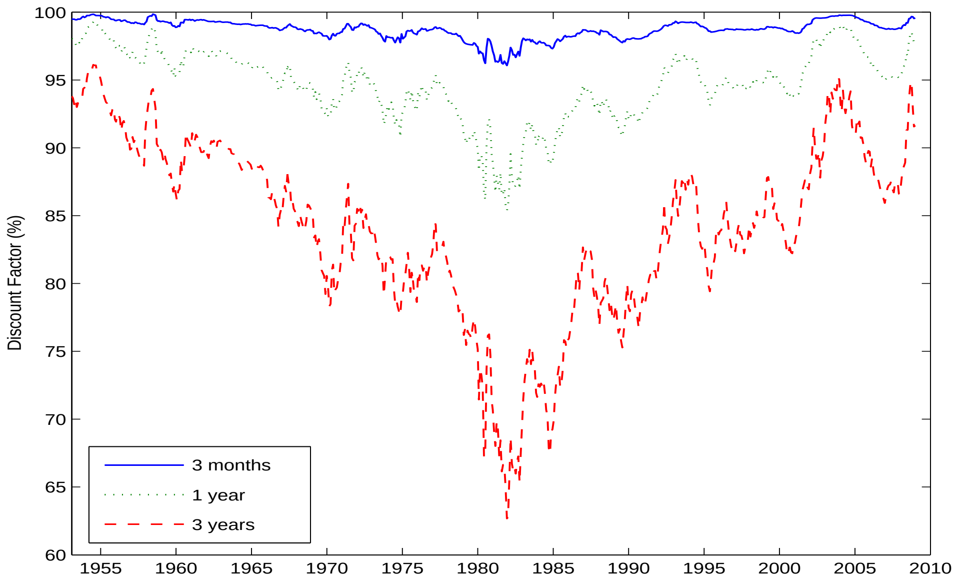
Figure 2.1 plot three discounts factors over time, from January 1953 to June 2008.[2] The top solid line is the 3-months discount factor, the middle dotted line is the 1-year discount factor, and bottom dashed line is the 3-year discount factor. First, note that indeed on each date in the sample, the discount factor with shorter time to maturity is always higher than the discount factor with longer time to maturity. Second, the variation of discount factors over time is rather substantial. For instance, the 3-year discount factor is as low as 0.6267 in August 1981, and as high as 0.95 in June 1954 and in June 2003.

Why do discount factors vary over time? Although this is a topic of a later chapter, it is useful to provide here the most obvious, and intuitive, reason. Figure 2.2 plots the time series of expected inflation from 1953 to 2008.[3] Comparing the discount factor series

[2]Data excerpted from CRSP (Fama Bliss discount bonds) ©2009 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. We discuss methodologies to estimate discount factors from bond data in Section 2.4.2 and in the Appendix.

[3]The expected inflation series is computed as the predicted annual inflation rate resulting from a rolling regression of inflation on its 12 lags. We present more details in Chapter 7.

Figure 2.1 Discount Factors



Source: Center for Research in Security Prices (CRSP)

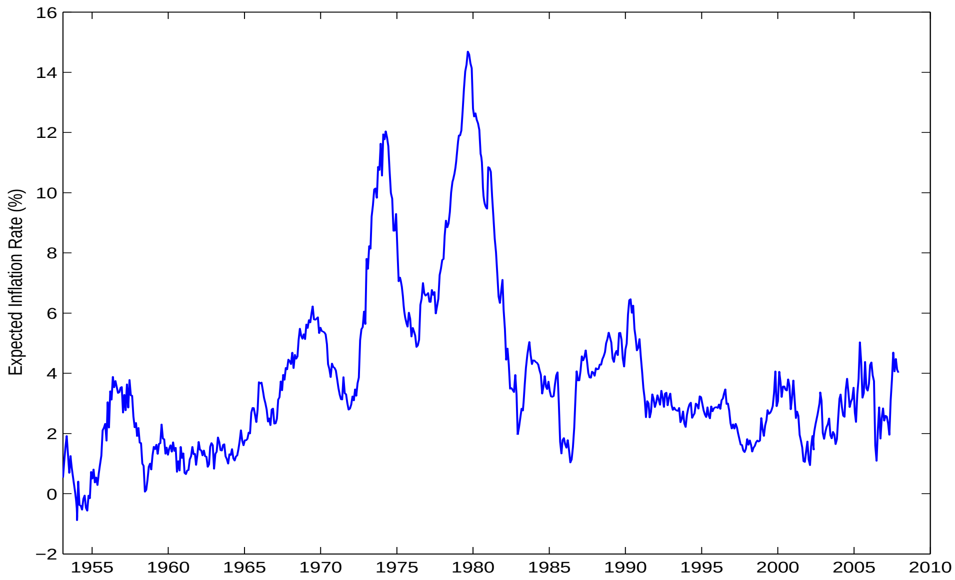
plotted in Figure 2.1 with the expected annual inflation series in Figure 2.2 it appears that expected inflation is an important determinant of discount factors. The intuition is also quite straightforward: Inflation is exactly what determines the time value of money, as it determines how much goods money can buy. The higher the expected inflation, the less appealing it is to receive money in the future compared to today, as this money will be able to buy a lesser amount of goods.

Although expected inflation is the most obvious culprit in explaining the variation over time of discount factors, it is not the only one. In Chapter 7 we look at various explanations that economists put forward to account for the behavior of discount factors and interest rates. These explanations are related to the behavior of the U.S. economy, its budget deficit, and the actions of the Federal Reserve, as well as investors' appetite for risk (or lack thereof). These macro economic conditions affect the relative supply and demand of Treasury securities and thus their prices.

2.2 INTEREST RATES

Grasping the concept of a rate of interest is both easier and more complicated than absorbing the concept of a discount factor. It is easier because the idea of interest is closer to our everyday notion of a return on an investment, or the cost of a loan. For instance, if we invest \$100 for one year at the rate of interest of 5%, we receive in one year \$105, that is, the original capital invested plus the interest on the investment. The same investment strategy could be described in terms of a discount factor as well: The discount factor here is the

Figure 2.2 Expected Inflation



Data Source: Bureau of Labor Statistics.

exchange rate between having \$105 in one year or \$100 today, that is 0.9524 = \$100/\$105. This latter number, which is equivalent to the 5% rate of interest, perhaps less intuitively describes the return on an investment.

The concept of an interest rate, however, is also more complicated, because it depends on the compounding frequency of the interest paid on the initial investment. The compounding frequency is defined as follows:

Definition 2.2 *The* **compounding frequency of interest accruals** *refers to the annual number of times in which interest is paid and reinvested on the invested capital.*

Indeed, to some extent, mentioning only an interest rate level is an incomplete description of the rate of return of an investment, or the cost of a loan or mortgage. The compounding frequency is a crucial element that must be attached to the interest rate figure.

For instance, in the above example, we implicitly assumed that the 5% rate of interest is applied to the original capital only once (and hence the $\$105$ result). However, if interest accrues, say, every 6 months, then the correct amount at maturity would be

$$(\$100) \times \left(1 + \frac{5\%}{2}\right) \times \left(1 + \frac{5\%}{2}\right) = \$105.0625,$$

which is higher than $\$105$. If interest accrues every month, then the correct amount at maturity would be

$$(\$100) \times \left(1 + \frac{5\%}{12}\right) \times \ldots [\text{ 12 times }] \ldots \times \left(1 + \frac{5\%}{12}\right) = \$105.11,$$

which is higher still. This example shows the following:

Fact 2.2 *For a given interest rate figure (e.g.* 5%*), more frequent accrual of interest yields a higher final payoff.*

Looking at interest rates from a different perspective, consider an investment in a security that costs \$100 today and that pays \$105 in one year, as in our earlier example. What is the rate of interest on this security? The intuitive answer is 5%, because we invest \$100 and obtain \$105, and thus the return equals $5\% = (\$105 - \$100)/\$100$. However, once again, the correct answer really depends on the frequency with which interest accrues on the security. If the interest accrues once a year, then 5% is the correct answer. However, if the interest accrues every half a year, for instance, the correct answer is given by $r = 4.939\%$. In fact, $(\$100) \times (1 + 4.939\%/2) \times (1 + 4.939\%/2) = \$105$, which is indeed the payoff from the investment of \$100. Similarly, if the interest accrues every month, then the correct answer to the same question is given by $r = 4.89\%$, as $(\$100) \times (1 + 4.89\%/12) \times \ldots [\text{ 12 times }] \ldots \times (1 + 4.89\%/12) = \$105$.

Fact 2.3 *For a given final payoff, more frequent accrual of interest implies a lower interest rate figure.*

This discussion emphasizes also the crucial difference between a *rate of return* on an investment, and an *interest rate*, which are related, but different, concepts. The rate of return is indeed the difference between payoff and initial investment, divided by the latter. In the example, $5\% = (\$105 - \$100)/\$100$ is the rate of return on the investment. The rate of interest corresponds instead to the (annualized) rate of return on the investment *within the compounding period*, but it differs from it otherwise. For instance, if the rate of interest is 5% and it accrues semi-annually, then within a six-month period the rate of return on the investment is 2.5%, that is, from \$100 we have in six months \$102.5. If we annualize this semi-annual return we obtain 5%, which corresponds to the rate of interest. Note, however, the rate of interest and the rate of return differ for a one-year horizon. In one year, the original investment will pay \$105.0625, as we obtained earlier, and thus the rate of return is 5.0625%$>$5%. When the horizon is longer, the discrepancy between the annualized interest rate figure and the annualized rate of return on the investment is larger.

2.2.1 Discount Factors, Interest Rates, and Compounding Frequencies

The examples above illustrate that discount factors and interest rates are intimately related, once we make explicit the compounding frequency. Given an interest rate and its compounding frequency, we can define a discount factor. Vice versa, given a discount factor, we can define an interest rate together with its compounding frequency. In this section we make the relation explicit.

Two compounding frequencies are particularly important: semi-annual compounding and continuous compounding. The semi-annual compounding frequency is the standard benchmark, as it matches the frequency of coupon payments of U.S. Treasury notes and bonds. The continuous compounding, defined below, is also important, mainly for its analytical convenience. As we shall see, formulas and derivations are much simpler under the assumption that the interest on an investment accrues infinitely frequently. This is of course an abstraction, but a useful one.

2.2.1.1 Semi-annual Compounding Let's begin with an example:

EXAMPLE 2.3

Let t = August 10, 2006, and let T = August 10, 2007 (one year later). Consider an investment of $\$100$ at t at the semi-annually compounded interest $r = 5\%$, for one year. As mentioned earlier, this terminology means that after six months the investment grows to $\$102.5 = \$100 \times (1 + 5\%/2)$, which is then reinvested at the same rate for another six months, yielding at T the payoff:

$$\text{Payoff at } T = \$105.0625 = (\$100) \times (1 + r/2) \times (1 + r/2) = (\$100) \times (1 + r/2)^2$$

Given that the initial investment is $\$100$, there are no cash flows to the investor during the period, and the payoff at T is risk free, the relation between money at t (\$100) and money at T ($= \$105.0625 =$ payoff at T) establishes a discount factor between the two dates, given by

$$Z(t,T) = \frac{\$100}{\text{payoff at } T} = \frac{1}{(1 + r/2)^2}$$

This example underlies the following more general statement:

Fact 2.4 *Let $r_2(t,T)$ denote the (annualized) semi-annually compounded interest rate between t and T. Then $r_2(t,T)$ defines a discount factor as*

$$Z(t,T) = \frac{1}{\left(1 + \frac{r_2(t,T)}{2}\right)^{2\times(T-t)}} \tag{2.2}$$

The logic of this fact lies in the example above. The semi-annually compounded interest rate $r_2(t,T)$ defines a payoff at maturity T given by

$$\text{Payoff at } T = \text{ Investment at } t \times \left(1 + \frac{r_2(t,T)}{2}\right)^{2\times(T-t)}.$$

Since the payoff at T is known at t, the relation between investment today at t and the payoff at T defines the time value of money, and $Z(t,T)$ given in Equation 2.2 defines the rate of exchange between money at T and money at t.

Similarly, given a discount factor $Z(t,T)$, we can obtain the semiannually compounded interest rate. The following example illustrates the point.

EXAMPLE 2.4

On March 1, 2001 (time t) the Treasury issued a 52-week Treasury bill, with maturity date T = February 28, 2002. The price of the Treasury bill was \$95.713. As we have learned, this price defines a discount factor between the two dates of $Z(t,T) = 0.95713$. At the same time, it also defines a semi-annually compounded interest rate equal to $r_2(t,T) = 4.43\%$. In fact, $\$95.713 \times (1 + 4.43\%/2)^2 = \$100$. The semi-annually compounded interest rate can be computed from $Z(t,T) = 0.95713$ by solving for $r_2(t,T)$ in Equation 2.2:

$$r_2(t,T) = 2 \times \left(\frac{1}{Z(t,T)^{\frac{1}{2}}} - 1 \right) = 2 \times \left(\frac{1}{0.95713^{\frac{1}{2}}} - 1 \right) = 4.43\% \qquad (2.3)$$

Fact 2.5 *Let $Z(t,T)$ be the discount factor between dates t and T. Then the semi-annually compounded interest rate $r_2(t,T)$ can be computed from the formula*

$$r_2(t,T) = 2 \times \left(\frac{1}{Z(t,T)^{\frac{1}{2 \times (T-t)}}} - 1 \right) \qquad (2.4)$$

2.2.1.2 More Frequent Compounding Market participants' time value of money – the discount factor $Z(0,T)$ – can be exploited to determine the interest rates with any compounding frequency, as well as the relation that must exist between any two interest rates which differ in compounding frequency. More precisely, if we let n denote the number of compounding periods per year (e.g., $n = 2$ corresponds to semi-annual compounding), we obtain the following:

Fact 2.6 *Let the discount factor $Z(t,T)$ be given, and let $r_n(t,T)$ denote the (annualized) n-times compounded interest rate. Then $r_n(t,T)$ is defined by the equation*

$$Z(t,T) = \frac{1}{\left(1 + \frac{r_n(t,T)}{n}\right)^{n \times (T-t)}} \qquad (2.5)$$

Solving for $r_n(t,T)$, we obtain

$$r_n(t,T) = n \times \left(\frac{1}{Z(t,T)^{\frac{1}{n \times (T-t)}}} - 1 \right) \qquad (2.6)$$

For instance, a \$100 investment at the monthly compounded interest rate $r_{12}(0,1) = 5\%$ yields by definition

$$\text{Payoff at } T = \$100 \times \left(1 + \frac{r_{12}(0,1)}{12}\right) \times \cdots [\text{ 12 times }] \cdots \times \left(1 + \frac{r_{12}(0,1)}{12}\right) = \$105.1162$$

Thus, the monthly compounded interest rate $r_{12}(0,1) = 5\%$ corresponds to the discount factor $Z(0,1) = \$100/\$105.1162 = 0.95133$, and vice versa.

2.2.1.3 Continuous Compounding. The continuously compounded interest rate is obtained by increasing the compounding frequency n to infinity. For all practical purposes, however, daily compounding – the standard for bank accounts – closely matches the continuous compounding, as we see in the next example.

Table 2.1 Interest Rate and Compounding Frequency

Compounding Frequency	n	$r_n(t, t+1)$
Annual	1	5.000%
Semi-annual	2	4.939%
Monthly	12	4.889%
Bi-monthly	24	4.883%
Weekely	52	4.881%
Bi-weekly	104	4.880%
Daily	365	4.879%
Bi-daily	730	4.879%
Hourly	8760	4.879%
Continuous	∞	4.879%

EXAMPLE 2.5

Consider the earlier example in which at t we invest $\$100$ to receive $\$105$ one year later. Recall that the annually compounded interest rate is $r_1(t, t+1) = 5\%$, the semi-annually compounded interest rate is $r_2(t, t+1) = 4.939\%$, and the monthly compounded interest rate is $r_{12}(t, t+1) = 4.889\%$. Table 2.1 reports the $n-$times compounded interest rate also for more frequent compounding. As it can be seen, if we keep increasing n, the $n-$ times compounded interest rate $r_n(t, t+1)$ keeps decreasing, but at an increasingly lower rate. Eventually, it converges to a number, namely, 4.879%. This is the continosly compounded interest rate. Note that in this example, there is no difference between the daily compounded interest rate ($n = 252$) and the one obtained with higher frequency ($n > 252$). That is, we can mentally think of continuous compounding as the daily compounding frequency.

Mathematically, we can express the limit of $r_n(t, T)$ in Equation 2.6 as n increases to infinity in terms of the exponential function:

Fact 2.7 *The* **continuously compounded** *interest rate $r(t, T)$, obtained from $r_n(t, T)$ for n that increases to infinity, is given by the formula*

$$Z(t, T) = e^{-r(t,T)(T-t)} \tag{2.7}$$

Solving for $r(t, T)$, we obtain

$$r\,(t, T) = -\frac{\ln\left(Z(t, T)\right)}{T - t} \tag{2.8}$$

where "$\ln(.)$" denotes the natural logarithm.

Returning to Example 2.5, we can verify Equation 2.8 by taking the natural logarithm of $Z(t, T) = \$100/\$105 = .952381$ and thus obtaining

$$r(t, T) = -\frac{\ln(Z(t, t+1))}{1} = 4.879\%$$

2.2.2 The Relation between Discounts Factors and Interest Rates

The previous formulas shows that given a discount factor between t and T, $Z(t,T)$, we can define interest rates of *any* compounding frequency by using Equations 2.2, 2.5, or 2.7. This fact implies that we can move from one compounding frequency to another by using the equalities implicit in these equations. For instance, for given interest rate $r_n(t,T)$ with n compounding frequency, we can determine the continuously compounded interest rate $r(t,T)$ by solving the equation

$$e^{-r(t,T)(T-t)} = Z(t,T) = \frac{1}{\left(1+\frac{r_n(t,T)}{n}\right)^{n\times(T-t)}} \tag{2.9}$$

Because of its analytical convenience, in this text we mostly use the continuously compounded interest rate in the description of discount factors, and for other quantities. Translating such number into another compounding frequency is immediate from Equation 2.9, which, more explicitly, implies

$$r(t,T) = n\times\ln\left(1+\frac{r_n\,(t,T)}{n}\right) \tag{2.10}$$

$$r_n(t,T) = n\times\left(e^{\frac{r(t,T)}{n}}-1\right) \tag{2.11}$$

To conclude, then, this section shows that the time value of money can be expressed equivalently through a discount factor, or in terms of an interest rate with its appropriate compounding frequency. At times, it will be convenient to focus on discount factors and at other times on interest rates, depending on the exercise. We should always keep in mind that the two quantities are equivalent.

2.3 THE TERM STRUCTURE OF INTEREST RATES

In the previous sections we noted that the primitive of our analysis is the discount factor, from which we define interest rates of various compounding frequencies. Interest rates, though, have a big advantage over discount factors when we analyze the time value of money: their units can be made uniform across maturities by annualizing them. The following example illustrates the point.

■ **EXAMPLE 2.6**

On June 5, 2008, the Treasury issued 13-week, 26-week and 52-week bills at prices $\$99.5399$, $\$99.0142$, and $\$97.8716$, respectively. Denoting $t =$ June 5, 2008, and T_1, T_2, and T_3 the three maturity dates, the implied discount factors are $Z(t,T_1) = 0.995399$, $Z(t,T_2) = 0.990142$, and $Z(t,T_3) = 0.978716$. The discount factor of longer maturities is lower than the one of shorter maturities, as Fact 2.1 would imply. The question is then: How much lower is $Z(t,T_3)$, say, compared to $Z(t,T_2)$ or $Z(t,T_1)$? Translating the discount factors into annualized interest rates provides a better sense of the relative value of money across maturities. In this case, the continuously compounded interest rates are

$$r(t,T_1) \quad = \quad -\frac{\ln(0.995399)}{0.25} = 1.8444\%;$$

$$r(t,T_2) = -\frac{\ln(0.990142)}{0.5} = 1.9814\%;$$
$$r(t,T_3) = -\frac{\ln(0.978716)}{1} = 2.1514\%.$$

The time value of money rises with maturity: The compensation that the Treasury has to provide investors to make them part with money today to receive money in the future, i.e., hold Treasury securities, increases the longer the investment period.

The term structure of interest rates is defined as follows:

Definition 2.3 *The* **term structure of interest rates***, or* **spot rate curve***, or* **yield curve***, at a certain time t defines the relation between the level of interest rates and their time to maturity $T-t$. The* **discount curve** *at a certain time t defines instead the relation between the discount factors $Z(t,T)$ and their time to maturity $T-t$.*

Figure 2.3 provides examples of spot curves $r(t,T)$ at four different dates.[4] These dates have been chosen also because the spot curves had different "shapes." Traders refer to these different shapes with particular names, which we now describe.[5]

Panel A of Figure 2.3 plots the term structure of interest rates on October 30, 1992. On the horizontal axis we have time to maturity $m = T - t$ for m that ranges from 3 months ($m = 0.25$) to 10 years ($m = 10$) (the letter "m" stands for "maturity"). The vertical axis represents the interest rate level $r(t, t+m)$ that corresponds to the various maturities. As can be seen, the term structure of interest rates on October 30, 1992 was increasing, which is a typical pattern in United States. The difference between the 10-year interest rate and the short-term interest rate is about 4%. This difference is called the term spread, or slope, of the term structure of interest rates.

Definition 2.4 *The* **term spread***, or* **slope***, is the difference between long-term interest rates (e.g. 10-year rate) and the short-term interest rates (e.g. 3-month rate).*

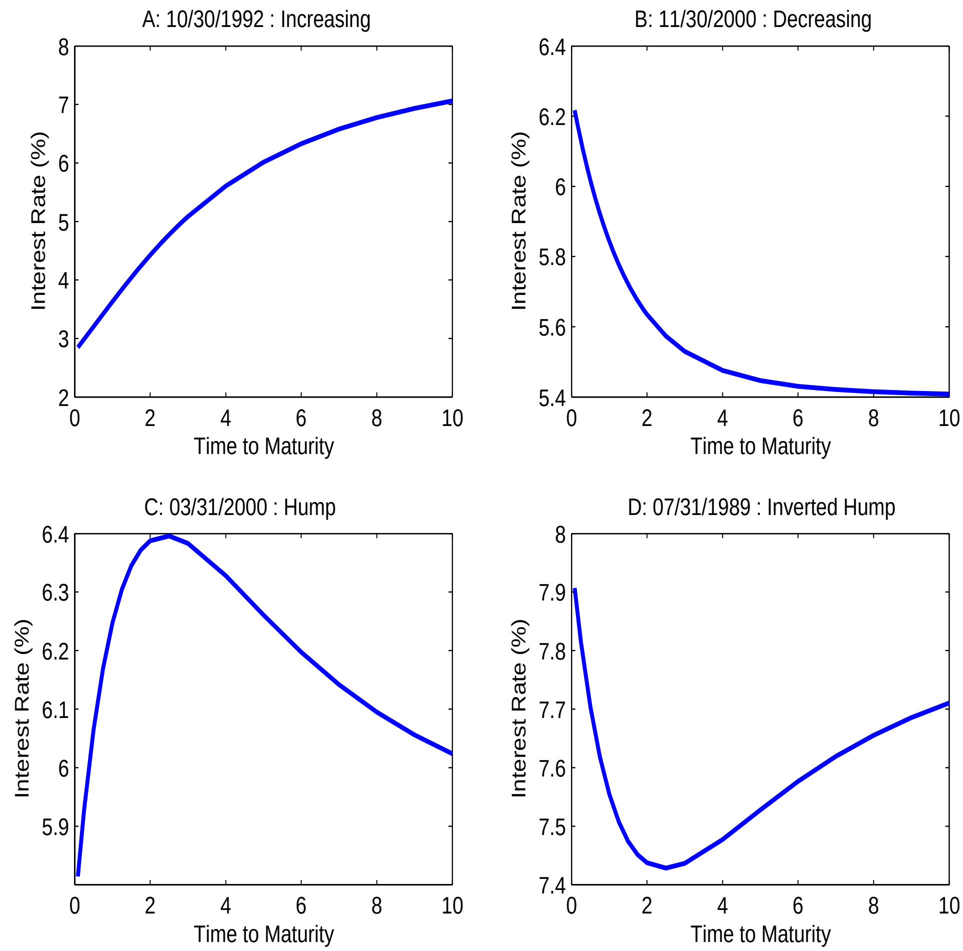
Typically, in the U.S. the term spread is positive. How is the term spread determined? Like discount factors, the term spread depends on numerous variables, such as expected future inflation, expected future growth of the economy, agents' attitudes toward risk, and so on. It is worth mentioning that although the expectation of future higher interest rates may determine today's term structure of interest rates, this is not the only channel. We will discuss more precisely the determinants of the term structure of interest rates in later chapters.

The shape of the term structure of interest rates is not always increasing. Panels B - D of Figure 2.3 plot the shape of the term structure on other occasions. In particular, Panel B illustrates a *decreasing* term structure of interest rates, as occurred on November 30, 2000. Panel C plots a term structure that is first rising and then decreasing. This shape is called *hump* and, in the example, took place on March 31, 2000. Finally, Panel D plots a term

[4]We calculated the spot curves using the extended Nelson Siegel model (see Section 2.9.3 in the appendix at the end of the chapter) based on data from CRSP (Monthly Treasuries) ©2009 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business.

[5]We are using the continuously compounded interest rate $r(t,T)$ to describe the curve. This is arbitrary. We could use any other compounding frequency, but as mentioned earlier, the continuously compounded frequency has some analytical advantages, as we shall see.

Figure 2.3 The Shapes of the Term Structure



Data Source: Center for Research in Security Prices

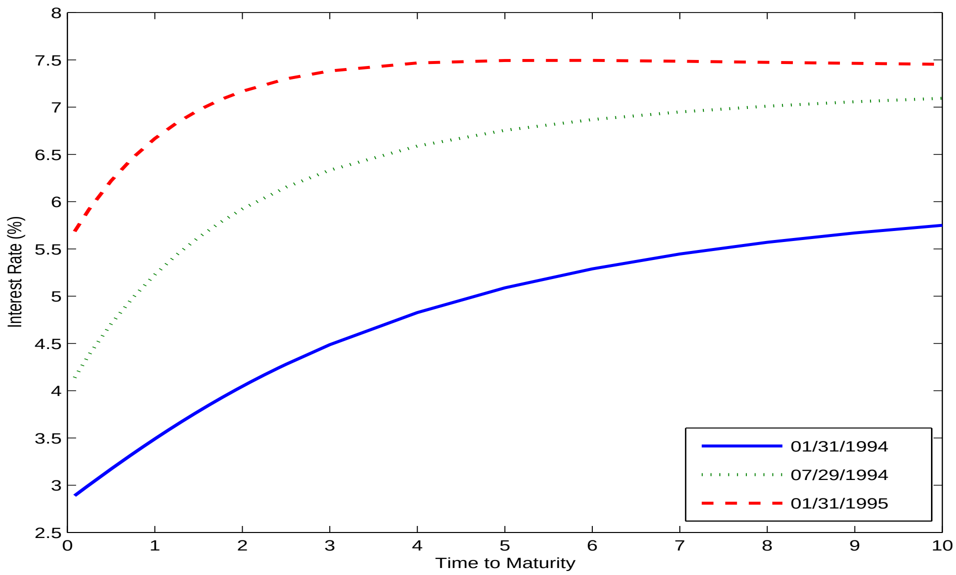
structure that is first decreasing and then increasing. This shape is called an *inverted hump*. The example in the Panel D is for July 31, 1989.

2.3.1 The Term Structure of Interest Rates over Time

As for discount factors, the term structure of interest rates depends on the date t at which it is computed. This is evident in Figure 2.3, as on three different dates we have three different shapes of the term structure. In particular, the dates corresponding to Panel C and Panel B of Figure 2.3 are only eight months apart, and yet the term structures are quite different in shape. In addition, besides the change in shape, the term structure also moves up and down as a whole.

Different dates correspond to different term structure of interest rates. For instance, Figure 2.4 plots the term structure of interest rates on three different dates, at six-months intervals, namely, from January 31, 1994 to January 31, 1995. In all three cases, the term structure of interest rates is increasing, but it is clear that it is lower and steeper for the first date, while it is higher and flatter for third date. The increase in the term structure is rather substantial: The short-term rate passed from 2.9% to 5.5%, while the long rate passed from 5.7% to 7.5%. This large change in the term structure of interest rates may have a devastating effect on the value of portfolios heavily invested in fixed income instruments. As we discuss in this and Chapter 3, it is exactly in 1994 that Orange County, a rich county in California, lost $1.6 billion and went bankrupt. The unexpected hike in interest rates together with an aggressive leveraged investment portfolio were the main causes of the debacle.

Figure 2.4 The Term Structure of Interest Rates on Three dates



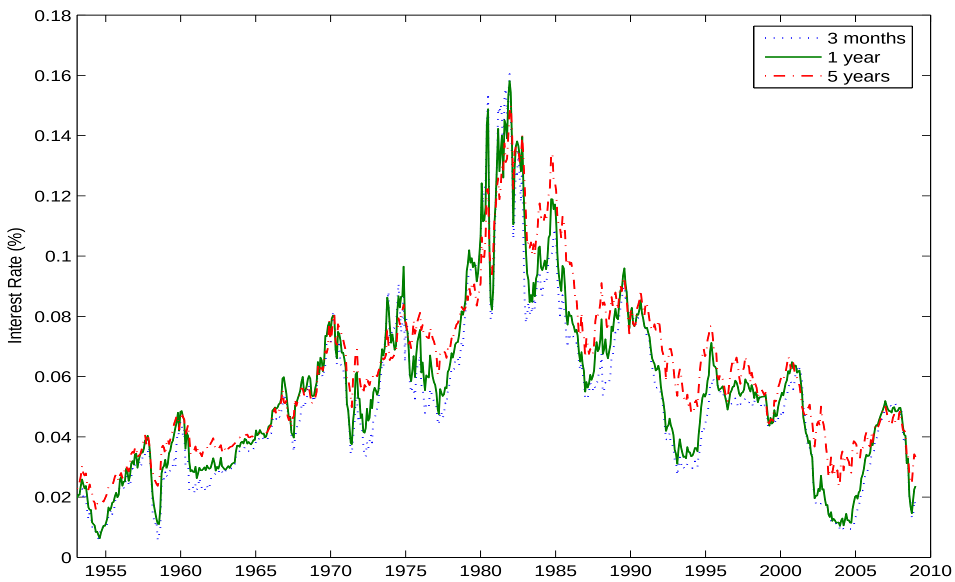
Data Source: Center for Research in Security Prices

As we did for discount factors, we can plot various points of the term structure of interest rates over time. Figure 2.5 graphs the term structure with maturities 3 months, 1 year, and 5 years from 1965 to 2008.[6] The first pattern that we see in Figure 2.5 is that interest rates move up and down substantially. The second clear pattern is that all of the interest rates move together: They go up and down at roughly the same time. However, it is also clear that they do not move up and down by the same amounts. To see this, let us consider the dotted line at the bottom of the plot, and the dashed line at the top. They corresponds to the 3-month and 5-year interest rates, respectively. As can be seen, they both go up and down

[6]Data excerpted from CRSP (Fama Bliss Discount Bonds and Fama Risk Free Rate) ©2009 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business.

at roughly the same time. We can also see that the term spread, the distance between the two rates, changes over time as well. For instance, focusing on the last part of the period, 2000 - 2006, we see that while the 3-month rate moved from over 6% down to 1%, then went back up to 5%, the 5-year rate moved from 6% down to 3%, and then back up to 5%. That is, the spread was zero both at the beginning and at the end of this sample, but it was large in the middle.

Figure 2.5 The Term Structure over Time



Source: Center for Research in Security Prices

Why does the term spread change over time? Once again, there are numerous reasons that contribute to the variation of both interest rates and term spreads, such as fluctuations in expected inflation, expected economic growth, and risk attitude of investors. We will review some recent theories in Chapter 7

2.4 COUPON BONDS

U.S. government Treasury bills involve only one payment from the Treasury to the investor at maturity. That is, their coupon rate is zero. They are thus an example of zero coupon bonds, bonds with no intermediate cash flows between issue date and maturity. The knowledge of the prices of zero coupon bonds allow us to determine the discount factor $Z(t,T)$, as described in the previous sections. More specifically, a government zero coupon bond at time t with maturity T has a price equal to

$$P_z(t,T) = 100 \times Z(t,T) \tag{2.12}$$

The subscript "z" is a mnemonic term for "zero" in zero coupon bond.

The Treasury issues zero coupon bonds with maturities up to only 52 weeks. For longer maturities, the Treasury issues securities that carry a coupon, that is, they also pay a sequence of cash flows (the coupons) between issue date and maturity, in addition to the final principal. In particular, the U.S. government issues Treasury notes, which are fixed income securities with maturity up to 10 years; Treasury bonds, which have maturities up to 30 years; and TIPS (Treasury Inflation Protected Securities), which have coupons that are not constant, but rather are linked to a recent inflation rate figure. We will talk about TIPS more exhaustively in Chapter 7. For now, we only consider Treasury notes and bonds. For convenience, we refer to both types as coupon bonds.

2.4.1 From Zero Coupon Bonds to Coupon Bonds

In this section we establish a relation between the prices of zero coupon bonds and coupon bonds. This relation forms the basis of much of the analysis that follows in later chapters, and so it is particularly important.

First, note that a coupon bond can be represented by the sequence of its cash payments. For instance, the 4.375% Treasury note issued on January 3, 2006 and with maturity of December 31, 2007, pays a cash flow of $\$2.1875$ on June 30, 2006, December 31, 2006, and June 30, 2007, while it pays $\$102.1875$ on December 31, 2007. Given the sequence of cash flows, which are certain in the sense that the U.S. Government is extremely unlikely to default, we could compute the value of the bond itself if we knew the discount factors $Z(t,T)$ to apply to each of the four dates. In fact, we can discount each future cash flow using its own discount factor, and sum the results.

Fact 2.8 *Consider a coupon bond at time t with coupon rate c, maturity T and payment dates T_1, T_2,...,$T_n = T$. Let there be discount factors $Z(t,T_i)$ for each date T_i. Then the value of the coupon bond can be computed as*

$$
\begin{aligned}
P_c(t,T_n) &= \frac{c\times 100}{2}\times Z(t,T_1)+\frac{c\times 100}{2}\times Z(t,T_2)+\ldots \\
&\quad \ldots+\left(\frac{c\times 100}{2}+100\right)\times Z\left(t,T_n\right) \\
&= \frac{c\times 100}{2}\times\sum_{i=1}^{n} Z(t,T_i)+100\times Z(t,T_n) \qquad (2.13)\\
&= \frac{c}{2}\times\sum_{i=1}^{n} P_z(t,T_i)+P_z(t,T_n) \qquad (2.14)
\end{aligned}
$$

The subscript "c" is a mnemonic device for "coupon" in coupon bond. Formula 2.14 shows that the coupon bond can be considered as a portfolio of zero coupon bonds.

■ **EXAMPLE 2.7**

Consider the 2-year note issued on $t =$ January 3, 2006 discussed earlier. On this date, the 6-month, 1-year, 1.5-years, and 2-year discounts were $Z(t,t+0.5) = 0.97862$, $Z(t,t+1) = 0.95718$, $Z(t,t+1.5) = 0.936826$ and $Z(t,t+2) = 0.91707$.

Therefore, the price of the note on that date was

$$P_c(t,T_n) = \$2.1875 \times \sum_{i=1}^{4} Z(t, t+0.5\times i) + \$100 \times 0.91707 = \$99.997,$$

which was indeed the issue price at t.

We can also represent the value of the coupon bond by using the semi-annual interest rate $r_2(t,T_i)$, where T_i, $i=1,...,n$, are the coupon payment dates. This representation is derived from the basic one above, but it can be useful nonetheless to report it:

$$P_c(t,T_n) = \sum_{i=1}^{n} \frac{c/2\times 100}{\left(1+r_2\left(t,T_i\right)/2\right)^{2\times(T_i-t)}} + \frac{100}{\left(1+r_2\left(t,T_n\right)/2\right)^{2\times(T_n-t)}} \quad (2.15)$$

A useful fact is the following:

Fact 2.9 *Let the semi-annual discount rate be constant across maturities,* $r_2(t,T_i)=r_2$ *for every* T_i*. At issue date* $t=0$*, the price of a coupon bond with coupon rate equal to the constant semi-annual rate* $c=r_2$ *is equal to par.*

$$P_c(0,T_n) = \sum_{i=1}^{n-1} \frac{c/2\times 100}{\left(1+r_2/2\right)^{2\times T_i}} + \frac{100\times(1+c/2)}{\left(1+r_2/2\right)^{2\times T_n}} = 100 \quad (2.16)$$

To understand the above fact, consider a 1-year note. Then, we can write

$$\begin{aligned}
P_c(0,T_2) &= \frac{c/2\times 100}{1+r_2/2} + \frac{100\times(1+c/2)}{\left(1+r_2/2\right)^2} && (2.17)\\
&= \frac{c/2\times 100}{1+r_2/2} + \left(\frac{100}{1+r_2/2}\right)\times\left(\frac{1+c/2}{1+r_2/2}\right) && (2.18)\\
&= \frac{c/2\times 100}{1+r_2/2} + \left(\frac{100}{1+r_2/2}\right)\times 1 && (2.19)\\
&= \frac{100\times(1+c/2)}{1+r_2/2} && (2.20)\\
&= 100 && (2.21)
\end{aligned}$$

This argument can be extended to many periods. The intuition is that any additional periods increase the cash flow by $c/2$ while they also increases the discount rate by the same amount $r_2/2$. The two forces move in opposite directions (more cash flows imply higher prices, while the additional discount imply lower price).

2.4.1.1 A No Arbitrage Argument We can establish Equation 2.13 also by appealing to a no arbitrage argument. In well-functioning markets in which both the coupon bond $P_c(t,T_n)$ and the zero coupon bonds $P_z(t,T_i)$ are traded in the market, if Equation 2.13 did not hold, an arbitrageur could make large risk-free profits. For instance, if

$$P_c(t,T_n) < \frac{c}{2}\times P_z(t,T_1) + \frac{c}{2}\times P_z(t,T_2) + ... + \left(1+\frac{c}{2}\right)\times P_z\left(t,T_n\right) \quad (2.22)$$

then the arbitrageur can buy the coupon bond for $P_c(t, T_n)$ and sell immediately $c/2$ units of zero coupon bonds with maturities T_1, T_2,..,T_{n-1} and $(1 + c/2)$ of the zero coupon bond with maturity T_n. This strategy yields an inflow of money to the arbitrageur that is equal to the difference between the right-hand side and the left-hand side of Equation 2.22. At every other maturity T_i the arbitrageur has a zero net position, as he receives the coupon from the Treasury and turns it around to the investors to whom the arbitrageur sold the individual zero coupon bonds. We note that this reasoning is the one that stands behind the law of one price, introduced in Fact 1.1 in Chapter 1, the fact that securities with identical cash flows should have the same price. The following example further illustrates the concept.

■ **EXAMPLE 2.8**

In Example 2.7, suppose that the 2-year note was trading at $98. An arbitrageur could purchase, say, $98 million of the 2-year note, and sell $2.1875 million of the 6-month, 1-year and 1.5-year zero coupons, and $102.1875 million of the 2-year zero coupon bond. The total value of the zeros the arbitrageur sells is $99.997 million, realizing approximately $2 million. The strategy is risk free, because at each coupon date in the future, the arbitrageur receives $2.1875 million from the Treasury, which he simply turns over to the investors who bought the zero coupon bonds. Similarly, at maturity, the arbitrageur receives $102.1875 million from the Treasury, and again turns it around to the investors of the last coupon.

In well-functioning markets such arbitrage opportunities cannot last for long. Thus, Equation 2.13 should hold "most of the time". It may happen that due to lack of liquidity or trading, some arbitrage opportunities may be detectable in the relative pricing of zero coupon bonds, such as STRIPS, and coupon bonds. However, these arbitrage opportunities are seldom exploitable: As soon as an arbitrageur tries to set up an arbitrage like the one described above, prices move instantly and the profit vanishes. Because expert arbitrageurs know this fact, some apparent mispricing may persist in the market place. We will regard such situations as "noise", that is, a little imprecision in market prices due to liquidity or external factors that sometimes impede the smooth functioning of capital markets.

2.4.2 From Coupon Bonds to Zero Coupon Bonds

We can also go the other way around: If we have enough coupon bonds, we can compute the implicit value of zero coupon bonds from the prices of coupon bonds. Equation 2.13 be used to *estimate* the discount factors $Z(t, T)$ for every maturity. The following example illustrates the reasoning:

■ **EXAMPLE 2.9**

On t = June 30, 2005, the 6-month Treasury bill, expiring on T_1 = December 29, 2005, was trading at $\$98.3607$. On the same date, the 1 year to maturity, 2.75% Treasury note, was trading at $\$99.2343$. The maturity of the latter Treasury note is T_2 = June 30, 2006. Given Equation 2.13, we can write the value of the two securities as:[7]

$$P_{bill}(t,T_1) = \$98.3607 \quad = \quad \$100 \times Z(t,T_1) \tag{2.23}$$
$$P_{note}(t,T_2) = \$99.2343 \quad = \quad \$1.375 \times Z(t,T_1) + \$101.375 \times Z(t,T_2) \tag{2.24}$$

We have two equations in two unknowns [the discount factors $Z(t,T_1)$ and $Z(t,T_2)$]. As in Section 2.1, from the first equation we obtain the discount factor $Z(t,T_1) = \$98.3607/\$100 = 0.983607$. We can substitute this value into the second equation, and solve for $Z(t,T_2)$ to obtain:

$$Z(t,T_2) = \frac{\$99.2343 - \$1.375 \times Z(t,T_1)}{\$101.375} = \frac{\$99.2343 - \$1.375 \times 0.983607}{\$101.375} = 0.965542$$

The prices of coupon bonds, then, implicitly contain the information about the market time value of money. This procedure can be iterated forward to obtain additional terms.

EXAMPLE 2.10

On the same date, $t =$ June 30, 2005, the December 31, 2006 Treasury note, with coupon of 3%, was trading at \$99.1093. Denoting by $T_3 =$ December 31, 2006, the price of this note can be written as

$$P_{note}(t,T_3) = \$99.1093 \quad = \quad \$1.5 \times Z(t,T_1) + \$1.5 \times Z(t,T_2) + \$101.5 \times Z(t,T_3) \tag{2.25}$$

We already determined $Z(t,T_1) = 0.983607$ and $Z(t,T_2) = 0.965542$ in Example 2.9. In Equation 2.25 the only unknown element is $Z(t,T_3)$. This is one equation in one unknown, and so we can solve for the $Z(t,T_3)$ to obtain

$$\begin{aligned} Z(t,T_3) &= \frac{\$99.1093 - \$1.5 \times (Z(t,T_1) + Z(t,T_2))}{\$101.5} \\ &= \frac{\$99.1093 - \$1.5 \times (0.983607 + 0.965542)}{\$101.5} \\ &= 0.947641 \end{aligned}$$

If we have a sufficient amount of data, we can proceed in this fashion for every maturity, and obtain all of the discount factors $Z(t,T)$. This methodology is called the bootstrap methodology.

Definition 2.5 *Let t be a given date. Let there be n coupon bonds, with coupons c_i, maturities T_i and prices denoted by $P(t,T_i)$. Assume that maturities are at regular intervals of six months, that is, $T_1 = t + 0.5$ and $T_i = T_{i-1} + 0.5$. Then, the* **bootstrap methodology** *to estimate discount factors $Z(t,T_i)$ for every $i = 1,...,n$ is as follows:*

1. The first discount factor $Z(t,T_1)$ is given by

$$Z(t,T_1) = \frac{P_c(t,T_1)}{100 \times (1 + c_1/2)} \tag{2.26}$$

[7] Notice a little approximation in this computation: The T-note would pay its coupon on December 31, 2005, rather than December 29. We assume that both dates correspond, approximately, to T_1.

2. *Any other discount factor* $Z(t,T_i)$ *for* $i = 2, ..., n$ *is given by*

$$Z(t,T_i) \quad = \quad \frac{P_c(t,T_i) - c_i/2 \times 100 \times \left(\sum_{j=1}^{i-1} Z(t,T_j))\right)}{100 \times (1 + c_i/2)} \qquad (2.27)$$

This procedure is relatively simple to implement, as the example above shows. One of the issues, though, is that bond data at six-month intervals are not always available. Unfortunately, this procedure requires all of the bonds, because otherwise the iterative procedure stops and there is no way to keep going. The appendix at the end of this chapter reviews some other methodologies that are widely used to estimate the discount factors $Z(0,T)$ from coupon bonds.

2.4.3 Expected Return and the Yield to Maturity

How can we measure the expected return on an investment in Treasury securities? Assuming the investor will hold the bond until maturity, computing the expected return on an investment in a zero coupon bond is relatively straightforward, as the final payoff is known and there are no intermediate cash flows. Thus, quite immediately, we have

$$\text{Return on zero coupon bond} = \frac{1}{Z(t,T)} - 1 \qquad (2.28)$$

This is the return between t and T. It is customary to annualize this amount, so that

$$\text{Annualized return on zero coupon bond} = \left(\frac{1}{Z(t,T)}\right)^{\frac{1}{T-t}} - 1 \qquad (2.29)$$

This, of course, corresponds to the annually compounded yield on the zero coupon, as in Equation 2.6 for $n = 1$.

For coupon bonds it is more complicated. A popular measure of return on investment for coupon bonds is called yield to maturity, which is defined as follows:

Definition 2.6 *Let* $P_c(t,T)$ *be the price at time* t *of a Treasury bond with coupon* c *and maturity* T. *Let* T_i *denote the coupon payments times, for* $i = 1, ..., n$. *The semi-annually compounded* **yield to maturity**, *or* **internal rate of return**, *is defined as the constant rate* y *that makes the discounted present value of the bond future cash flows equal to its price. That is,* y *is defined by the equation*

$$P_c(t,T) = \sum_{i=1}^{n} \frac{c/2 \times 100}{(1 + y/2)^{2\times(T_i - t)}} + \frac{100}{(1 + y/2)^{2\times(T_n - t)}} \qquad (2.30)$$

Before moving to interpret this measure of return on investment, it is important to recognize a major distinction between the formula in Equation 2.30 and the one that we obtained earlier in terms of discount factors, namely Equation 2.15. Although they appear the same, it is crucial to note that the yield to maturity y is defined as the particular *constant* rate that makes the right-hand side of Equation 2.30 equal to the price of the bond. Instead, Equation 2.15 is the one defining the price of the bond from the discount factors $Z(t, T)$. Unless the term structure of interest rates is exactly flat, the yields at various maturities are different, and will not coincide with the yield to maturity y. Indeed, to some extent, the yield to maturity y can be considered an average of the semi-annually compounded spot rates $r_2(0, T)$, which define the discount $Z(0, T)$. However, it is important to note that this "average" depends on the coupon level c. In fact:

Fact 2.10 *Two different bonds that have the same maturity but different coupon rates c have different yield to maturities y.*

This fact is easily illustrated with an example:

EXAMPLE 2.11

Columns 1 to 6 of Table 2.2 display coupon rates, maturities, and quotes of the latest issued Treasury notes on February 15, 2008.[8] Column 7 shows the discount curve $Z(0, T)$ obtained from the bootstrap procedure discussed in Section 2.4.2, and Column 8 reports the continuously compounded spot rate curve $r(0, T)$.

On February 15, 2008, traders could buy or sell two Treasury securities with the same maturity $T = 9.5$ years, but with very different coupon rates. In particular, a T-note with coupon $c = 4.750\%$ and a T-bond with coupon $c = 8.875\%$ were available. Using the discount factors $Z(0, T)$ in Table 2.2 and the formula in Equation 2.15 we can determine the fair prices of the two securities. In particular, we have[9]

$$\begin{aligned}\text{Price T-note}_{\ c=4.750} &= \frac{4.750}{2} \times \left(\sum_{T=0.5}^{9.5} Z(0, T) \right) + 100 \times Z(0, 9.5) \\ &= 107.8906 \qquad (2.31)\\ \text{Price T-bond}_{\ c=8.875} &= \frac{8.875}{2} \times \left(\sum_{T=0.5}^{9.5} Z(0, T) \right) + 100 \times Z(0, 9.5) \\ &= 141.5267 \qquad (2.32)\end{aligned}$$

What are the yield to maturity of these two securities? Solving Equation 2.30 for the two bonds, the yield to maturity of the $c = 4.75$ T-note and $c = 8.875$ T-bond are, respectively

$$y_{c=4.750} = 3.7548\%. \qquad (2.33)$$

$$y_{c=8.875} = 3.6603\% \qquad (2.34)$$

As it can be seen from Equations 2.33 and 2.34, the bond with the higher coupon has lower yield to maturity y.

[8] Data excerpted from CRSP (Daily Treasuries) ©2009 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business.

[9] We use fair prices, i.e., prices obtained from the same discount curve $Z(0, T)$, to better illustrate the concept of yield to maturity and its relation to a bond coupon rate. It turns out however that on February 15, 2008, the 9.5-year T-bond with $c = 8.875$ was actually trading at 140.0781, about 1% less than its fair price computed in Equation 2.32. This lower price is due to the lack of liquidity of bonds that have been issued long in the past compared to the latest issued T-notes used to compute the discount curve $Z(0, T)$.

Table 2.2 Term Structure on February 15, 2008

Coupon Rate (%)	Maturity Date	Time to Maturity	Bid	Ask	Mid	Bootstrap Discount	Spot Curve	Discount with $y = 3.7548\%$	Discount with $y = 3.6603\%$
4.125	8/15/2008	0.5	100.9844	101.0156	101.0000	98.9590	2.0930	98.1572	98.2027
4.500	2/15/2009	1.0	102.6094	102.6406	102.6250	98.1892	1.8274	96.3484	96.4378
4.875	8/15/2009	1.5	104.4766	104.5078	104.4922	97.3147	1.8147	94.5729	94.7045
4.750	2/15/2010	2.0	105.5078	105.5391	105.5234	96.2441	1.9141	92.8301	93.0024
4.125	8/15/2010	2.5	105.0859	105.1172	105.1016	95.0822	2.0172	91.1194	91.3309
5.000	2/15/2011	3.0	108.2344	108.2656	108.2500	93.7612	2.1473	89.4403	89.6895
5.000	8/15/2011	3.5	109.0000	109.0313	109.0156	92.2213	2.3137	87.7920	88.0775
4.875	2/15/2012	4.0	109.1719	109.2031	109.1875	90.6046	2.4666	86.1742	86.4945
4.375	8/15/2012	4.5	107.3281	107.3594	107.3438	88.7259	2.6582	84.5862	84.9400
3.875	2/15/2013	5.0	105.1406	105.1719	105.1563	86.9809	2.7896	83.0274	83.4134
4.250	8/15/2013	5.5	106.8125	106.8438	106.8281	85.0858	2.9365	81.4974	81.9142
4.000	2/15/2014	6.0	105.2344	105.2656	105.2500	83.1241	3.0806	79.9956	80.4420
4.250	8/15/2014	6.5	106.3281	106.3594	106.3438	81.1114	3.2207	78.5214	78.9962
4.000	2/15/2015	7.0	104.3750	104.4063	104.3906	79.0613	3.3564	77.0744	77.5765
4.250	8/15/2015	7.5	105.4063	105.4375	105.4219	76.8759	3.5064	75.6541	76.1822
4.500	2/15/2016	8.0	106.7188	106.7500	106.7344	74.8256	3.6251	74.2600	74.8130
4.875	8/15/2016	8.5	109.0000	109.0313	109.0156	72.6763	3.7548	72.8915	73.4684
4.625	2/15/2017	9.0	106.9375	106.9688	106.9531	70.8392	3.8306	71.5483	72.1480
4.750	8/15/2017	9.5	107.8750	107.9063	107.8906	69.1582	3.8818	70.2298	70.8513
3.500	2/15/2018	10.0	97.8750	97.9063	97.8906	68.1581	3.8334	68.9356	69.5779

Data excerpted from CRSP (Daily Treasuries) ©2009 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business.

To verify the yield to maturity computed are indeed correct, Table 2.2 also reports the discount factors $Z^y(0,T) = (1+y/2)^{-2\times T}$ implied by Equations 2.33 and 2.34. Using these two discounts $Z^y(0,T)$ instead of $Z(0,T)$ in Equations 2.31 and 2.32, respectively, we indeed obtain the correct prices:

$$\text{Price T-note}_{c=4.750} = \sum_{T=0.5}^{9.5} \frac{4.750/2}{\left(1+\frac{3.7548\%}{2}\right)^{2\times T}} + \frac{100}{\left(1+\frac{3.7548\%}{2}\right)^{2\times 9.5}} = 107.8906$$

$$\text{Price T-bond}_{c=8.875} = \sum_{T=0.5}^{9.5} \frac{8.875/2}{\left(1+\frac{3.6603\%}{2}\right)^{2\times T}} + \frac{100}{\left(1+\frac{3.6603\%}{2}\right)^{2\times 9.5}} = 141.5267$$

This example shows that there is something curious in the definition of yield to maturity. Why does the coupon rate affect the yield to maturity? To understand the intuition, we need to note that y correctly measures the expected return on an investment only under the strict condition that *the investor can reinvest all of the coupons at the rate y over the life of the bond.*

To see this, let us compute the total payoff at maturity T assuming that the investor can reinvest all of the coupons paid at dates T_1, T_2, ..., T_{n-1} at the constant rate y for the remaining periods $T - T_1$, $T - T_2$, ..., $T - T_{n-1}$. This is given by the following:

$$\text{Total payoff at } T = \frac{c \times 100}{2} \times (1 + y/2)^{2\times(T-T_1)}$$

$$
\begin{aligned}
& +\frac{c \times 100}{2} \times (1+y/2)^{2\times(T-T_2)} \\
& \dots \\
& +\frac{c \times 100}{2} \times (1+y/2)^{2\times(T-T_{n-1})} \\
& +\frac{c \times 100}{2} + 100 \\
= \quad & \frac{c \times 100}{2} \times \left(\sum_{i=1}^{n} (1+y/2)^{2\times(T-T_i)}\right) + 100
\end{aligned}
$$

We can now compute the present value of the total payoff at T, using y as the constant semi-annual yield and thus $Z^y(t,T) = (1+y/2)^{-2(T-t)}$ as the discount factor. This gives

$$
\begin{aligned}
\begin{array}{c}\text{Present value of}\\ \text{total payoff at } T\end{array} \quad &= \quad Z^y(t,T) \times \left[\frac{c \times 100}{2} \times \left(\sum_{i=1}^{n} (1+y/2)^{2\times(T-T_i)}\right) + 100\right] \\
&= \quad \frac{c \times 100}{2} \times \left(\sum_{i=1}^{n} \frac{(1+y/2)^{2\times(T-T_i)}}{(1+y/2)^{2\times(T-t)}}\right) + \frac{100}{(1+y/2)^{2\times(T-t)}} \\
&= \quad \frac{c \times 100}{2} \times \left(\sum_{i=1}^{n} \frac{1}{(1+y/2)^{2\times(T_i-t)}}\right) + \frac{100}{(1+y/2)^{2\times(T-t)}} \\
&= \quad P(t,T)
\end{aligned}
$$

We find then that the price of the bond is (by definition) equal to the present value of the total payoff at T, discounted at the yield to maturity y, under the assumption that all of the coupons can be reinvested at same rate y over the life the of the bond.

Given that it is practically impossible for an investor to reinvest all of the coupons at the constant yield to maturity y , this latter measure is in fact a poor measure of expected return. Indeed, the definition of a return on an investment cannot be given without a precise definition of the time interval during which the security is held. For instance, a 10-year STRIP provides the *certain* annualized return in Equation 2.29 *if* the security is held until maturity. However, if the investor sells the bond after one year, the return may be higher or lower than the promised yield (Equation 2.29) depending on what happens to interest rates. A substantial increase in interest rates, for instance, will tend to lower prices of long-term bonds, and thus the investor can end up with a capital loss. In later chapters we use modern financial concepts to precisely define the *expected* return on an investment during a given period, as well as the no arbitrage restrictions that must exist across bonds.

Why then do traders use the notion of yield to maturity y in their every day trading? Given a coupon rate c, Equation 2.30 shows that there is a one-to-one relation between the price P_c of the bond and the yield to maturity y. Thus, a trader can quote the same bond by using P_c or y. To some extent, then, the yield to maturity is just a convenient way of quoting a bond price to other traders.

2.4.4 Quoting Conventions

We end this section on Treasury bonds with a few remarks on the market quoting convention for Treasury bills and Treasury bonds.

2.4.4.1 Treasury Bills. Treasury bills are quoted on a discount basis. That is, rather than quoting a price $P_{bill}(t,T)$ for a Treasury bill, Treasury dealers quote the following quantity

$$d = \frac{100 - P_{bill}(t,T)}{100} \times \frac{360}{n} \tag{2.35}$$

where n is the number of calendar days between t and T. For instance, on August 10, 2006 the Treasury issued a 182-day bill at a price of \$97.477 for \$100 of face value. Treasury dealers quoted this price as

$$d = \frac{100 - 97.477}{100} \times \frac{360}{182} = 4.99\%$$

where d is an annualized discount rate on the face value of the Treasury bill.

Give a quote d from a Treasury dealer, we can compute the price of the Treasury bill by solving for $P_{bill}(t,T)$ in Equation 2.35:

$$P_{bill}(t,T) = 100 \times \left[1 - \frac{n}{360} \times d\right] \tag{2.36}$$

2.4.4.2 Treasury Coupon Notes and Bonds. Coupon notes and bonds present an additional complication. Between coupon dates, interest accrues on the bond. If a bond is purchased between coupon dates, the buyer is only entitled to the portion of the coupon that accrues between the purchase date and the next coupon date. The seller of the bond is entitled to the portion of the coupon that accrued between the last coupon and the purchase date. It is market convention to quote Treasury notes and bonds without any inclusion of accrued interests. However, the buyer agrees to pay the seller any accrued interest between the last coupon date and purchase price. That is, we have the formula

$$\text{Invoice price} = \text{Quoted price} + \text{Accrued interest} \tag{2.37}$$

The quoted price is sometimes referred to as the **clean price** while the invoice price is sometimes also called **dirty price**.

The accrued interest is computed using the following intuitive formula:

$$\begin{aligned} \text{Accrued interest} &= \text{Interest due in the full period} \times \\ &\times \frac{\text{Number of days since last coupon date}}{\text{Number of days between coupon payments}} \end{aligned}$$

Market conventions also determine the methodology to count days. There are three main ways:

1. Actual/Actual: Simply count the number of calendar days between coupons;
2. 30/360: Assume there are 30 days in a month and 360 in a year;
3. Actual/360: Each month has the right number of days according to the calendar, but there are only 360 days in a year.

Which convention is used depends on the security considered. For instance, Treasury bills use actual/360 while Treasury notes and bonds use the actual/actual counting convention.

2.5 FLOATING RATE BONDS

Floating rate bonds are coupon bonds whose coupons are tied to some reference interest rate. The U.S. Treasury does not issue floating rate bonds, but governments of other countries as well as individual corporations do. We present an example of floating rate bond in Example 2.12. It is important to spend some time on the pricing of floating rate bonds as a similar methodology applies to numerous other interest rate securities, such as floaters and inverse floaters (see Case Study in Section 2.8) as well as derivative instruments, such as interest rate swaps (see Chapter 5).[10]

■ **EXAMPLE 2.12**

The Italian Treasury regularly issues CCT (Certificati di Credito del Tesoro), which are floating rate bonds with 7 years to maturity. The CCT semi-annual coupon is equal to the most recent rate on the six-month BOT (the Italian Treasury bill), plus a spread (fixed at 0.15%). There is a six-month temporal lag between the determination of the coupon and its actual payment.

Unless otherwise specified, we therefore define a floating rate bond as follows:

Definition 2.7 *A semi-annual* **floating rate bond** *with maturity T is a bond whose coupon payments at dates $t = 0.5$, 1, ..., T are determined by the formula*

$$\text{Coupon payment at } t = c(t) = 100 \times (r_2\,(t - 0.5) + s) \tag{2.38}$$

where $r_2(t)$ is the 6-month Treasury rate at t, and s is a spread.[11] *Each coupon date is also called* **reset date** *as it is the time when the new coupon is reset (according to the formula).*

2.5.1 The Pricing of Floating Rate Bonds

The pricing of floating rate bonds is simple, although the logic may appear a little complicated at first. Consider the case in which the spread $s = 0$. In this case, the following is true

Fact 2.11 *If the spread $s = 0$, the ex-coupon price of a floating rate bond on any coupon date is equal to the bond par value.*[12]

To understand the logic, consider first the following simple example

■ **EXAMPLE 2.13**

[10] In this chapter we only review the pricing of floating rate bonds for the case in which the coupon rate is linked to the same interest rate that is also used for discounting purposes, which grealy simplifies the analysis and provides the formulas needed for future applications.
[11] For notational simplicity, in this section the six month rate is denoted by $r_2(t)$ instead of $r_2(t, t + 0.5)$.
[12] Ex-coupon means that the price does not incorporate the coupon that is paid on that particular day. Par value is the bond's principal amount.

Consider a one year, semi-annual floating rate bond. The coupon at time $t = 0.5$ depends on *today*'s interest rate $r_2(0)$, which is known. If today $r_2(0) = 2\%$, then $c(0.5) = 100 \times 2\%/2 = 1$. What about the coupon $c(1)$ at maturity $T = 1$? This coupon will depend on the 6-month rate at time $t = 0.5$, which we do not know today. This implies that we do not know the value of the final cash flow at time $T = 1$, which is equal to $100 + c(1)$. Computing the present value of this uncertain cash flow initially seems hard. And yet, with a moment's reflection, it is actually simple. Consider an investor who is evaluating this bond. This investor can project himself to time $t = 0.5$, six months before maturity. Can the investor at time $t = 0.5$ guess what the cash flow will be at time $T = 1$? Yes, because at time $t = 0.5$ the investor will know the interest rate. So, he can compute what the value is at time $t = 0.5$. Suppose that at time $t = 0.5$ the interest rate is $r_2(0.5) = 3\%$, then the coupon at time $T = 1$ is $c(1) = 100 \times r_2(0.5)/2 = 1.5$. This implies that the value of the bond at time $t = 0.5$ is equal to

$$\text{Value bond at } 0.5 = \text{Present value of } (100 + c(1)) = \frac{100 + 1.5}{1 + 0.03/2} = 100,$$

which is a round number, equal to par. What if the interest rate at time $t = 0.5$ was $r_2(0.5) = 6\%$? In this case, the coupon rate at time $T = 1$ is $c(1) = 100 \times r_2(0.5)/2 = 103$, and the value of the bond at $t = 0.5$ is

$$\text{Value bond at } 0.5 = \text{Present value of } (100 + c(1)) = \frac{100 + 3}{1 + 0.06/2} = 100$$

Still the same round number, equal to par. Indeed, *independently* of the level of the interest rate $r_2(T_1)$, we find that the value of bond at $t = 0.5$ is always equal to 100:

$$\begin{aligned} \text{Value bond at } 0.5 &= \text{Present value of } (100 + c(1)) = \frac{100 + 100 \times r_2(0.5)/2}{1 + r_2(0.5)/2} \\ &= \frac{100 \times (1 + r_2(0.5)/2)}{1 + r_2(0.5)/2} \\ &= 100. \end{aligned}$$

Even if the investor does not know the cash flow at time $T = 1$, because it depends on the future floating rate $r_2(0.5)$, the investor *does* know that at time $t = 0.5$ the ex-coupon value of the floating rate bond *will be* 100, independently of what the interest rate does then. But then, he can compute the value of the bond at time $t = 0$, because the coupon at time $T_1 = 0.5$ is known at time $t = 0$ as it is given by $c(0.5) = 100 \times r_2(0)/2 = 101$. Thus, the value at time $t = 0$ is

$$\text{Value bond at } 0 \quad = \quad \text{Present value of } (100 + c(0.5)) = \frac{100 + 1}{1 + .02/2} = 100$$

The result that at time $t = 0.5$ the ex-coupon bond price of this floating rate bond is always equal to 100 may appear puzzling, but it is actually intuitive. When the interest rate move from $r_2(0) = 2\%$ to $r_2(0.5)$ it has two effects:

1. It changes the future cash flow $c(1) = 100 \times r_2(0.5)$

- If the interest rate $r_2(0.5)$ rises, the future cash flow increases.
- If the interest rate $r_2(0.5)$ declines, the future cash flow declines.

2. It changes the discount rate to apply to the future cash flows.

- If the interest rate $r_2(0,5)$ rises, the discount rate increases.
- If the interest rate $r_2(0,5)$ declines, the discount rate declines.

The two effects, called "cash flow effect" and "discount effect," work in opposite directions. If the interest rate increases, the future cash flow increases, but it is discounted by a higher rate.

The institutional feature of lagging the coupon payment by six months allows for this cash flow and discount effect to exactly cancel each other out, leaving the value of the bond at 100 at any reset date.

What if there are additional dates? The reasoning is the same, and we work backwards. Table 2.3 contains the description of computations for the valuation of a 2-year floating rate bond. Briefly, starting from the top of the table, the ex-coupon value at $T = 2$ is simply the principal 100. The cum-coupon value is principal times the semi-annual interest rate $r_2(1.5)/2$ determined six months earlier, at $t = 1.5$. We can compute the present value as of $t = 1.5$ of the total cash flow at time $T = 2$, resulting in the ex-coupon price still equal to 100. The logic is the same as in Example 2.13. The cum-coupon price at $t = 1.5$ is then 100 plus the coupon, which equals 100 times the semi-annual rate determined at $t = 1$. Thus, the present value as of $t = 1$ of the total cash flow at time $t = 1.5$ (i.e., coupon plus value of bond at $t = 1.5$) is equal to 100, again. And so on until $t = 0$.

2.5.2 Complications

We must discuss two simplifying assumptions made above: First, the spread s on the floating rate is zero. Second, the time 0 of the valuation is a reset date. Fortunately, the generalization to the more realistic case is simple.

First, if the spread s is nonzero we can decompose the total cash flow per period in two parts, the floating part and the fixed part. This decomposition results in the equality

$$\text{Floating coupon with spread } s = \text{Floating coupon with zero spread} + \text{Fixed coupon } s$$

We can then value independently each component on the right-hand side, as we already know how to value a floating coupon bond with zero spread (see previous section) and a sequence of fixed coupon payments equal to s. Indeed, the present value of the fixed sequence of payments equal to s is $\sum_{t=0.5}^{T} s \times Z(0,t)$. Therefore, we have the equality:

$$\begin{aligned}\text{Price floating rate bond with spread } s \quad = \quad & \text{Price floating rate bond with zero spread} \\ & + s \times \sum_{t=0.5}^{n} Z(0,t)\end{aligned}$$

Table 2.3 The Valuation of a 2-year Floating Rate Bond

Time	Rate	Coupon	Ex-Coupon Price	Cum-Coupon Price
2	$r_2(2)$	$c(2) = 100 \times \frac{r_2(1.5)}{2}$	$P_{FR}(2) = 100$	$P_{FR}^C(2) = 100 + c(2) = 100 \times \left(1 + \frac{r_2(1.5)}{2}\right)$
1.5	$r_2(1.5)$	$c(1.5) = 100 \times \frac{r_2(1)}{2}$	$P_{FR}(1.5) = \frac{P_{FR}^C(2)}{1+\frac{r_2(1.5)}{2}} = \frac{100\times\left(1+\frac{r_2(1.5)}{2}\right)}{1+\frac{r_2(1.5)}{2}} = 100$	$P_{FR}^C(1.5) = P_{FR}(1.5) + c(1.5) = 100 \times \left(1 + \frac{r_2(1)}{2}\right)$
1	$r_2(1)$	$c(1) = 100 \times \frac{r_2(0.5)}{2}$	$P_{FR}(1) = \frac{P_{FR}^C(1.5)}{1+\frac{r_2(1)}{2}} = \frac{100\times\left(1+\frac{r_2(1)}{2}\right)}{1+\frac{r_2(1)}{2}} = 100$	$P_{FR}^C(1) = P_{FR}(1) + c(1) = 100 \times \left(1 + \frac{r_2(0.5)}{2}\right)$
0.5	$r_2(0.5)$	$c(0.5) = 100 \times \frac{r_2(0)}{2}$	$P_{FR}(0.5) = \frac{P_{FR}^C(1)}{1+\frac{r_2(0.5)}{2}} = \frac{100\times\left(1+\frac{r_2(0.5)}{2}\right)}{1+\frac{r_2(0.5)}{2}} = 100$	$P_{FR}^C(0.5) = P_{FR}(0.5) + c(0.5) = 100 \times \left(1 + \frac{r_2(0)}{2}\right)$
0	$r_2(0)$	-	$P_{FR}(0) = \frac{P_{FR}^C(0.5)}{1+\frac{r_2(0)}{2}} = \frac{100\times\left(1+\frac{r_2(0)}{2}\right)}{1+\frac{r_2(0)}{2}} = 100$	

At reset dates the price of the floating rate bond with zero spread is just par (=100), so that [13]

$$\text{Price floating rate bond with spread } s \quad = \quad 100 + s \times \sum_{t=0.5}^{T} Z(0,t) \qquad (2.39)$$

The second complication is that the valuation may be outside reset dates. Consider first Example 2.13. If today is not $t = 0$, but $t = 0.25$, how do we value the floating rate bond? The backward induction argument up to $t = 0.5$ still holds: At time $t = 0.5$ the ex-coupon bond price will be worth 100 and the cum-coupon bond price will be worth $\$101 = \$100 + \$100 \times 2\%/2$. The only difference from before is that we now have to discount the amount 101 not back to $t = 0$ at the rate $r_2(0) = 2\%$, but back to $t = 0.25$ at the current 3-month rate. For instance, if the quarterly compounded 3-months rate is also 2%, that is $r_4(0.25, 0.5) = 2\%$, then

$$\text{Value bond at } 0.25 = \text{Present value of } \$101 = \frac{\$101}{(1 + 0.02/4)} = \$100.4975$$

In this case, the value of the bond depends on the current interest rate. If for instance $r_4(0.25, 0.5) = 1\%$, the value of the bond at 0.25 is \$100.7481.

The same reasoning applies more generally. Let us denote by 0 the last reset date, and by t the current trading day. Then, we know that at the next reset date, time 0.5, the ex-coupon value of the floating rate bond will be \$100. Thus, the cum-coupon value of the floating rate bond at the next reset date is

$$P^C_{FR}(0.5, T) = 100 + c(0.5) = 100 + 100 \times r_2(0)/2$$

Therefore, the value at time $0 < t < 0.5$ of this cash flow is

$$P_{FR}(t, T) = \text{Present value of } P^C_{FR}(0.5, T) = Z(t, 0.5) \times 100 \times [1 + r_2(0)/2]$$

We summarize these results in the following:

Fact 2.12 *Let T_1, T_2, ... T_n be the floating rate reset dates and let the current date t be between time T_i and T_{i+1}: $T_i < t < T_{i+1}$. The general formula for a semi-annual floating rate bond with zero spread s is*

$$P_{FR}(t, T) = Z(t, T_{i+1}) \times 100 \times [1 + r_2(T_i)/2] \qquad (2.40)$$

where $Z(t, T_{i+1})$ is the discount factor from t to T_{i+1}. At reset dates, $Z(T_i, T_{i+1}) = 1/(1 + r_2(T_i)/2)$, which implies

$$P_{FR}(T_i, T) = 100 \qquad (2.41)$$

It may be useful to note that although between coupon dates the value of a floating rate bond depends on the interest rates, its sensitivity to variation in interest rate is very small, as we shall see more fully in later chapters.

[13] The spread s often reflects a lower credit quality than the reference rate used. The appropriate discount factors should then be used to discount future cash flows. For instance, if the reference rate is the LIBOR, then the LIBOR curve should be used, as discussed in Chapter 5.

2.6 SUMMARY

In this chapter, we covered the following topics:

1. Discount factors: The discount factor is the value today of one dollar in the future. Discount factors decrease with the time horizon and also vary over time.

2. Interest rates: The promised rate of return of an investment, an interest rate needs a compounding frequency to be well defined. They are quoted on annualized basis.

3. Compounding frequency: This is the frequency with which interest on an investment is accrues over time. Continuous compounding refers to the limit in which payments accrue every instant. Practically, daily compounding is very close to continuous compounding.

4. Term structure of interest rates: The term structure of interest rate is the relation between the interest rates and maturity. Investment horizons affect the interest rate to be received on an investment or paid on a loan.

5. Zero coupon bonds: These are securities that pay only one given amount (par) at maturity. Examples are Treasury bills or STRIPS.

6. Coupon notes and bonds. These are securities that pay a sequence of coupons and the principal back at maturity. Examples are Treasury notes and bonds, which pay coupon semi-annually. T-notes have maturity up to ten years, while bonds up to thirty years.

7. Bootstrap: This procedure is for computing discount factors at various maturities from data on coupon notes and bonds. It requires the availability of notes and bonds at semi-annual intervals.

2.7 EXERCISES

1. Figure 2.3 shows that the term structure of interest rates can be declining, with short-term spot rates higher than long-term spot rates. How steep can the decline in spot rates be? Consider two STRIPS: One has 3-years to maturity and yields a continuously compounded rate $r(0,3) = 10\%$, while the second has 5 years to maturity and yields a continuously compounded rate $r(0,5) = 5\%$. Discuss whether this scenario is possible, and, if not, what arbitrage strategy could be set up to gain from the mispricing.

2. Compute the price, the yield and the continuously compounded yield for the following Treasury bills. For the 1-year Treasury bill also compute the semi-annually compounded yield.

   (a) 4-week with 3.48% discount (December 12, 2005)

   (b) 4-week with 0.13% discount (November 6, 2008)

   (c) 3-month with 4.93% discount (July 10, 2006)

   (d) 3-month with 4.76% discount (May 8, 2007)

(e) 3-month with 0.48% discount (November 4, 2008)

(f) 6-month with 4.72% discount (April 21, 2006)

(g) 6-month with 4.75% discount (June 6, 2007)

(h) 6-month with 0.89% discount (November 11, 2008)

(i) 1-year with 1.73% discount (September 30, 2008)

(j) 1-year with 1.19% discount (November 5, 2008)

3. You are given the following data on different rates with the same maturity (1.5 years), but quoted on a different basis and different compounding frequencies:

   - Continuously compounded rate: 2.00% annualized rate
   - Continuously compounded return on maturity: 3.00%
   - Annually compounded rate: 2.10% annualized rate
   - Semi-annually compounded rate: 2.01% annualized rate

   You want to find an arbitrage opportunity among these rates. Is there any one that seems to be mispriced?

4. Using the semi-annually compounded yield curve in Table 2.4, price the following securities:

   (a) 5-year zero coupon bond

   (b) 7-year coupon bond paying 15% semiannually

   (c) 4-year coupon bond paying 7% quarterly

   (d) 3 1/4-year coupon bond paying 9% semiannually

   (e) 4-year floating rate bond with zero spread and semiannual payments

   (f) 2 1/2-year floating rate bond with zero spread and annual payments

   (g) 5 1/2-year floating rate bond with 35 basis point spread with quarterly payments

   (h) 7 1/4-year floating rate bond with 40 basis point spread with semiannual payments

5. Consider a 10-year coupon bond paying 6% coupon rate.

   (a) What is its price if its yield to maturity is 6%? What if it is 5% or 7%?

   (b) Compute the price of the coupon bond for yields ranging between 1% and 15%. Plot the resulting bond price versus the yield to maturity. What does the plot look like?

6. Consider the data in Table 2.4. Consider two bonds, both with 7 years to maturity, but with different coupon rates. Let the two coupon rates be 15% and 3%.

   (a) Compute the prices and the yields to maturity of these coupon bonds.

   (b) How do the yields to maturity compare to each other? If they are different, why are they different? Would the difference in yields imply that one is a better "buy" than the other?

Table 2.4 Yield Curve on March 15, 2000

Maturity	Yield	Maturity	Yield	Maturity	Yield
0.25	6.33%	2.75	6.86%	5.25	6.39%
0.50	6.49%	3.00	6.83%	5.50	6.31%
0.75	6.62%	3.25	6.80%	5.75	6.24%
1.00	6.71%	3.50	6.76%	6.00	6.15%
1.25	6.79%	3.75	6.72%	6.25	6.05%
1.50	6.84%	4.00	6.67%	6.50	5.94%
1.75	6.87%	4.25	6.62%	6.75	5.81%
2.00	6.88%	4.50	6.57%	7.00	5.67%
2.25	6.89%	4.75	6.51%	7.25	5.50%
2.50	6.88%	5.00	6.45%	7.50	5.31%

Yields calculated based on data from CRSP (Daily Treasuries).

7. Today is May 15, 2000.

   (a) Compute the discount curve $Z(0, T)$ for T =6 month, 1 year, 1.5 years, and 2 years from the following data:

   - A 6-month zero coupon bond priced at $96.80 (issued 5/15/2000)
   - A 1-year note with 5.75% coupon priced at $99.56 (issued 5/15/1998)
   - A 1.5 year note with 7.5% coupon priced at $100.86 (issued 11/15/1991)
   - A 2-year note with 7.5% coupon priced at $101.22 (issued 5/15/1992)

   (b) Once you get the discount curve $Z(0, T)$ you take another look at the data and you find the following 1-year bonds:

   i. A 1-year note with 8.00% coupon priced at $101.13 (issued 5/15/1991)
   ii. A 1-year bond with 13.13% coupon priced at $106.00 (issued 4/2/1981)

   Compute the prices for these bonds with the discounts you found. Are the prices the same as what the market says? Is there an arbitrage opportunity? Why?

8. On March 15, 2000 you obtain the data on Treasuries in Table 2.5. Compute the semiannual yield curve, spanning over 9 years, from the data using the bootstrap procedure.

9. The Orange County case study at the end of the chapter discusses the pricing of inverse floaters, and provides a decomposition of inverse floaters in terms of a coupon bond, a floating rate bond, and a zero coupon bond (see Equation 2.43). Find an alternative decomposition of the same security, and compute the price. Do you obtain the same price? Discuss your findings in light of the law of one price discussed in Chapter 1

2.8 CASE STUDY: ORANGE COUNTY INVERSE FLOATERS

Table 2.5 Bonds and Notes on March 15, 2000

Cusip	Issue Date	Maturity Date	Name	Coupon	Bid	Ask
912827ZE	8/15/1990	8/15/2000	NOTE	8.750%	100.5742	100.6055
912827ZN	11/15/1990	11/15/2000	NOTE	8.500%	100.8906	100.9219
912810CT	1/12/1981	2/15/2001	BOND	11.750%	103.8047	103.8359
912810CU	4/2/1981	5/15/2001	BOND	13.125%	105.9805	106.0117
912810CW	7/2/1981	8/15/2001	BOND	13.375%	107.6406	107.6719
912810CX	10/7/1981	11/15/2001	BOND	15.750%	112.3945	112.4258
912810CZ	1/6/1982	2/15/2002	BOND	14.250%	111.9297	111.9609
912827F4	5/15/1992	5/15/2002	NOTE	7.500%	101.2031	101.2344
912827G5	8/15/1992	8/15/2002	NOTE	6.375%	99.0469	99.0781
912810DA	9/29/1982	11/15/2002	BOND	11.625%	110.6680	110.6992
912810DC	1/4/1983	2/15/2003	BOND	10.750%	109.5117	109.5430
912810DD	4/4/1983	5/15/2003	BOND	10.750%	110.3281	110.3594
912810DE	7/5/1983	8/15/2003	BOND	11.125%	112.1523	112.1836
912810DG	10/5/1983	11/15/2003	BOND	11.875%	115.3086	115.3398
912827N8	2/15/1994	2/15/2004	NOTE	5.875%	97.1172	97.1484
912810DH	4/5/1984	5/15/2004	BOND	12.375%	118.8984	118.9297
912810DK	7/10/1984	8/15/2004	BOND	13.750%	124.9375	124.9688
912810DM	10/30/1984	11/15/2004	BOND	11.625%	118.2969	118.3281
912827S8	2/15/1995	2/15/2005	NOTE	7.500%	102.8633	102.8945
912810DQ	4/2/1985	5/15/2005	BOND	12.000%	121.6133	121.6445
912810DR	7/2/1985	8/15/2005	BOND	10.750%	117.0664	117.0977
912827V8	11/15/1995	11/15/2005	NOTE	5.875%	95.9844	96.0156
912810DU	1/15/1986	2/15/2006	BOND	9.375%	112.0352	112.0664
912827X8	5/15/1996	5/15/2006	NOTE	6.875%	100.6055	100.6367
912827Y5	7/15/1996	8/15/2006	NOTE	7.000%	101.2031	101.2344
912827Z6	10/15/1996	11/15/2006	NOTE	6.500%	98.7500	98.7813
9128272J	2/15/1997	2/15/2007	NOTE	6.250%	97.4883	97.5195
9128272U	5/15/1997	5/15/2007	NOTE	6.625%	99.5625	99.5938
9128273E	8/15/1997	8/15/2007	NOTE	6.125%	96.7578	96.7891
9128273X	2/15/1998	11/15/2007	NOTE	5.500%	93.1328	93.1641
9128274F	5/15/1998	5/15/2008	NOTE	5.625%	93.7852	93.8164
9128274V	11/16/1998	11/15/2008	NOTE	4.750%	87.9766	88.0078
9128275G	5/17/1999	5/15/2009	NOTE	5.500%	92.8242	92.8555

Data excerpted from CRSP (Daily Treasuries) ©2009 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business.

With the tools we have developed in this chapter we can price all sorts of securities.[14] An interesting security to price is an inverse floater. These securities became very popular during the Orange County bankruptcy at the end of 1994, since it is estimated that a significant fraction of the county's portfolio comprised these securities. The bankruptcy of Orange County is a classic example of the risk that is inherent in interest rate securities, and we will discuss this case study more thoroughly in Chapters 3 and 4 after we introduce some tools to measure interest rate risk. In this section we pave the way for the discussion of risk in the next chapters by investigating the methodology to value inverse floaters. Because we are interested in understanding the dynamics behind Orange County's financial problems, we assume that we are beginning our analysis on December 31, 1993 (a year before the county declared bankruptcy).

2.8.1 Decomposing Inverse Floaters into a Portfolio of Basic Securities

An inverse floater is a security that pays a lower coupon as interest rates go up (hence the name inverse floater). For this to work, we need to establish a fixed reference rate from which to subtract the floating rate. To keep things simple, we assume that the inverse floater promises to pay 15% minus the short-term interest rate on an annual basis with 3 years maturity. That is, the coupon on the bond is:

$$c(t) = 15\% - r_1(t-1) \tag{2.42}$$

where $r_1(t-1)$ denotes the annually compounded rate at time $t-1$, and we adopt the usual convention according to which the cash flow at time t, $c(t)$, depends on the interest rate one period earlier, namely $t-1$ in our case as payments are annual. The assumption of the annual payment frequency for the inverse floater is made for simplicity, so that the calculations are easier to follow. Notice also that Equation 2.42 contains a further simplification, namely, the fact that the coupon is always positive, which would be violated if the short rate were ever larger than 15%. In reality, if this situation occurs, the bond does not pay any coupon [i.e., $c(t) = 0$ if $r(t-1) > 15\%$]. To take this case into account we need to develop further tools, as we will do in Parts II and III of this book. For the time being assume that we know with certainty that the short rate is always below 15%.

An interesting fact arises by looking at the formula in Equation 2.42: Coupon payments are a combination of a fixed rate and a floating rate bond. So, for the coupon payments, this is the same as having a long position in a fixed rate bond and a short position in a floating rate bond, as such positions would entail receiving a fixed coupon and paying a floating coupon.[15] If we follow this strategy, however, we find that at maturity $T = 3$ the principal we receive from the fixed rate bond has to be used to pay for the principal of the floating rate bond. That is, only a long position in the fixed rate and short position in the floating rate bond does not exactly mimic an inverse floater. We can solve this problem by adding to the portfolio a 3-year zero coupon bond. From the law of one price (see Fact 1.1

[14]Thanks to Francisco Javier Madrid for his help in putting this case together. Descriptive material is from the case study ERISK: Orange County, downloaded from http://www.erisk.com/Learning/CaseStudies/OrangeCounty.asp.
[15]An investor has a long position in a bond if he holds the bond in the portfolio. In contrast, a short position means that the investor sold the bond without actually having it in the portfolio. The short position is accomplished by first borrowing the bond from a broker, typically in the repo market, and then selling it to the market. It is the responsability of the investor who sold the bond short to make the regular coupon payments to the counterparty.

in Chapter 1) the price of an inverse floater is then:

$$\text{Price inverse floater} = P_z(0,3) + P_c(0,3) - P_{FR}(0,3) \tag{2.43}$$

where we recall that $P_z(0,3)$, $P_c(0,3)$, and $P_{FR}(0,3)$ denote the prices of a zero coupon bond, a coupon bond, and a floating rate bond with three years to maturity.

2.8.2 Calculating the Term Structure of Interest Rates from Coupon Bonds

The next challenge is to determine the term structure of the interest rates so as to obtain the discount rates for the bonds. A first idea might be to find zero coupon bonds for all these periods. The problem is that we might not necessarily find all the data we want. An alternative is to look at all bonds that are being quoted in the market today (December 31, 1993) and use this data to plot the yield curve. The reasoning is that every day, quotes are available on bonds maturing at different dates. As we saw in this chapter, in absence of arbitrage opportunities, any bond with coupon c that matures in three years (even if it was issued, for example, seven years ago), must have the same price as a 3-year bond, issued today, with coupon c. Additionally, through Equations 2.26 and 2.27 we can convert coupon paying bonds into zero coupon bonds. This exercise, however, may prove more challenging than it sounds. Look at Table 2.6, which reports all the bond price quotes available on December 31, 1993. There are 224 bonds quoted. To perform the bootstrap analysis, we need many fewer bonds. How do we pick the bonds to bootstrap out the term structure of interest rates?

After some careful (and time consuming) analysis of the data in Table 2.6, we resolve to use the subsample of data, contained in Table 2.7, which are nicely spaced at 6-month intervals. The last two columns of the table provide the discount factors $Z(0,T)$ computed from either bid prices or ask prices.[16]

2.8.3 Calculating the Price of the Inverse Floater

Recall that the value of the inverse floater can be computed from the value of a zero coupon bond, a coupon bond with coupon rate equal to 15%, and a floating rate bond, all of them with maturity equal to three years. Given the discount factors in Table 2.7, we can obtain values for these standard bonds. For simplicity, we use the average discount $Z(0,T) = 0.5 \times Z_A(0,T) + 0.5 \times Z_B(0,T)$ for the following calculations.

1. **Three-year zero coupon bond**. Three years from December 31, 1993 corresponds to the maturity date December 31, 1996. The discount factor $Z(0,3) = 0.8745$. Thus, $P_z(0,3) = 100 \times 0.8745 = \$87.45$.

2. **Three-year, 15% fixed coupon bond**. Given the discount factors $Z(0,T)$ for $T = 1,2,3$, we can compute the price of a coupon bond by applying the bond pricing formula in Equation 2.13, with the only caveat that in this exercise coupons are annually paid, and thus we do not have to divide them by 2 (as instead we do in Equation 2.13). More precisely, Table 2.8 carries out the calculation, and obtains the price of the fixed coupon bond $P_c(0,T) = \$128.83$.

[16]Recall that the bid and ask prices are the quotes at which security dealers are ready to buy or sell the securities. Because they make a profit from the spread between them, the ask price is higher than the bid price.

Table 2.6 Bond Quotes on December 31, 1993

Maturity	Coupon	Bid	Ask	Maturity	Coupon	Bid	Ask	Maturity	Coupon	Bid	Ask
19940106	0.000	99.960	99.961	19950415	8.375	105.500	105.563	19980531	5.375	101.125	101.188
19940113	0.000	99.904	99.908	19950430	3.875	99.938	100.000	19980630	5.125	100.063	100.125
19940115	7.000	100.094	100.156	19950515	10.375	108.500	108.563	19980715	8.250	112.500	112.563
19940120	0.000	99.841	99.846	19950515	12.625	111.563	111.688	19980731	5.250	100.469	100.531
19940127	0.000	99.788	99.795	19950515	5.875	102.531	102.594	19980815	9.250	116.750	116.813
19940131	4.875	100.125	100.188	19950515	8.500	106.000	106.063	19980831	4.750	98.375	98.438
19940203	0.000	99.724	99.728	19950515	11.250	109.656	109.719	19980930	4.750	98.375	98.438
19940210	0.000	99.666	99.671	19950531	4.125	100.250	100.313	19981015	7.125	108.188	108.250
19940215	9.000	100.688	100.750	19950630	4.125	100.188	100.250	19981031	4.750	98.188	98.250
19940215	6.875	100.406	100.469	19950715	8.875	107.125	107.188	19981115	3.500	98.969	99.969
19940215	8.875	100.656	100.719	19950731	4.250	100.281	100.344	19981115	8.875	115.750	115.813
19940217	0.000	99.612	99.617	19950815	4.625	100.875	100.938	19981130	5.125	99.625	99.688
19940224	0.000	99.551	99.557	19950815	8.500	106.875	106.938	19981231	5.125	99.625	99.688
19940228	5.375	100.344	100.406	19950815	10.500	110.031	110.094	19990115	6.375	105.031	105.094
19940303	0.000	99.489	99.492	19950831	3.875	99.688	99.750	19990215	8.875	116.156	116.219
19940310	0.000	99.423	99.427	19950930	3.875	99.594	99.656	19990415	7.000	107.844	107.906
19940317	0.000	99.367	99.371	19951015	8.625	107.625	107.688	19990515	9.125	117.875	117.938
19940324	0.000	99.306	99.311	19951031	3.875	99.531	99.594	19990715	6.375	105.000	105.063
19940331	5.750	100.625	100.688	19951115	11.500	113.063	113.188	19990815	8.000	112.938	113.000
19940331	8.500	101.250	101.313	19951115	5.125	101.688	101.750	19991015	6.000	103.219	103.281
19940331	0.000	99.248	99.253	19951115	8.500	107.688	107.750	19991115	7.875	112.625	112.688
19940407	0.000	99.173	99.178	19951115	9.500	109.531	109.594	20000115	6.375	105.063	105.125
19940414	0.000	99.113	99.119	19951130	4.250	100.063	100.125	20000215	8.500	116.125	116.188
19940415	7.000	101.031	101.094	19951231	4.250	100.000	100.063	20000415	5.500	100.844	100.906
19940421	0.000	99.047	99.053	19960115	9.250	109.625	109.688	20000515	8.875	118.375	118.438
19940428	0.000	98.990	98.997	19960131	7.500	106.344	106.406	20000815	8.750	118.000	118.063
19940430	5.375	100.656	100.719	19960215	4.625	100.688	100.750	20001115	8.500	116.844	116.906
19940505	0.000	98.920	98.927	19960215	7.875	107.125	107.188	20010215	11.750	136.219	136.344
19940512	0.000	98.849	98.856	19960215	8.875	109.156	109.219	20010215	7.750	113.719	113.781
19940515	7.000	101.313	101.375	19960229	7.500	106.500	106.563	20010515	13.125	145.313	145.438
19940515	9.500	102.219	102.281	19960331	7.750	107.188	107.250	20010515	8.000	114.344	114.406
19940515	13.125	103.563	103.625	19960415	9.375	110.750	110.813	20010815	13.375	147.875	148.000
19940519	0.000	98.780	98.788	19960430	7.625	107.094	107.156	20010815	7.875	113.813	113.875
19940526	0.000	98.714	98.723	19960515	4.250	99.719	99.781	20011115	15.750	164.031	164.156
19940531	5.125	100.719	100.781	19960515	7.375	106.625	106.688	20011115	7.500	111.563	111.625
19940602	0.000	98.653	98.661	19960531	7.625	107.375	107.438	20020215	14.250	155.656	155.781
19940609	0.000	98.582	98.591	19960630	7.875	108.688	108.750	20020515	7.500	111.844	111.906
19940616	0.000	98.516	98.525	19960715	7.875	108.125	108.188	20020815	6.375	104.250	104.313
19940623	0.000	98.439	98.449	19960731	7.875	108.219	108.281	20021115	11.625	140.344	140.469
19940630	5.000	100.813	100.875	19960815	4.375	99.844	99.906	20030215	10.750	134.813	134.938
19940630	8.500	102.500	102.563	19960831	7.250	106.813	106.875	20030215	6.250	103.250	103.313
19940630	0.000	98.391	98.401	19960930	7.000	106.594	106.656	20030515	10.750	135.313	135.438
19940715	8.000	102.406	102.469	19961015	8.000	108.938	109.000	20030815	11.125	138.656	138.781
19940728	0.000	98.113	98.125	19961031	6.875	106.125	106.188	20030815	5.750	99.625	99.688
19940731	4.250	100.469	100.531	19961115	4.375	99.563	99.625	20031115	11.875	144.750	144.875
19940815	8.750	103.188	103.250	19961115	7.250	107.156	107.219	20040515	12.375	149.969	150.094
19940815	6.875	102.063	102.125	19961130	6.500	105.250	105.313	20040815	13.750	161.406	161.531
19940815	8.625	103.125	103.188	19961231	6.125	104.531	104.594	20041115	11.625	145.219	145.344
19940815	12.625	105.531	105.594	19970115	8.000	109.469	109.531	20050515	12.000	149.375	149.500
19940825	0.000	97.841	97.854	19970131	6.250	104.594	104.656	20050815	10.750	139.531	139.656
19940831	4.250	100.469	100.531	19970228	6.750	106.063	106.125	20060215	9.375	128.906	129.031
19940922	0.000	97.563	97.578	19970331	6.875	106.531	106.594	20150215	11.250	154.250	154.313
19940930	4.000	100.313	100.375	19970415	8.500	111.375	111.438	20150815	10.625	147.375	147.438
19940930	8.500	103.594	103.656	19970430	6.875	106.563	106.625	20151115	9.875	138.750	138.813
19941015	9.500	104.438	104.500	19970515	8.500	111.531	111.594	20160215	9.250	131.531	131.594
19941020	0.000	97.249	97.265	19970531	6.750	106.219	106.281	20160515	7.250	108.188	108.250
19941031	4.250	100.531	100.594	19970630	6.375	105.156	105.219	20161115	7.500	111.031	111.094
19941115	10.125	105.438	105.500	19970715	8.500	111.875	111.938	20170515	8.750	125.938	126.000
19941115	6.000	101.969	102.031	19970731	5.500	102.344	102.406	20170815	8.875	127.594	127.656
19941115	8.250	103.875	103.938	19970815	8.625	112.469	112.531	20180515	9.125	130.969	131.031
19941115	11.625	106.750	106.813	19970831	5.625	102.563	102.625	20181115	9.000	129.625	129.688
19941117	0.000	96.924	96.942	19970930	5.500	102.188	102.250	20190215	8.875	128.219	128.281
19941130	4.625	100.875	100.938	19971015	8.750	113.125	113.188	20190815	8.125	119.094	119.156
19941215	0.000	96.646	96.665	19971031	5.750	102.969	103.031	20200215	8.500	123.906	123.969
19941231	4.625	100.906	100.969	19971115	8.875	113.781	113.844	20200515	8.750	127.219	127.281
19941231	7.625	103.813	103.875	19971130	6.000	103.781	103.844	20200815	8.750	127.281	127.344
19950115	8.625	104.875	104.938	19971231	6.000	103.781	103.844	20210215	7.875	116.438	116.500
19950131	4.250	100.531	100.594	19980115	7.875	110.438	110.500	20210515	8.125	119.719	119.781
19950215	3.000	100.250	101.250	19980131	5.625	102.281	102.344	20210815	8.125	119.719	119.781
19950215	10.500	107.344	107.406	19980215	8.125	111.500	111.563	20211115	8.000	118.406	118.469
19950215	5.500	101.844	101.906	19980228	5.125	100.438	100.500	20220815	7.250	109.094	109.156
19950215	7.750	104.281	104.344	19980331	5.125	100.344	100.406	20221115	7.625	114.156	114.219
19950215	11.250	108.125	108.188	19980415	7.875	110.719	110.781	20230215	7.125	108.156	108.219
19950228	3.875	100.063	100.125	19980430	5.125	100.250	100.313	20230815	6.250	98.656	98.719
19950331	3.875	100.031	100.094	19980515	9.000	115.250	115.313				

Data excerpted from CRSP (Daily Treasuries) ©2009 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business.

Table 2.7 Discount Factors $Z(0,T)$ on December 31, 1993

Maturity	Coupon	Bid	Ask	$Z_B(0,T)$	$Z_A(0,T)$
19940630	0.000	98.3911	98.4012	0.9839	0.9840
19941231	7.625	103.8125	103.8750	0.9639	0.9645
19950630	4.125	100.1875	100.2500	0.9423	0.9429
19951231	4.250	100.0000	100.0625	0.9191	0.9196
19960630	7.875	108.6875	108.7500	0.9014	0.9019
19961231	6.125	104.5313	104.5938	0.8743	0.8748
19970630	6.375	105.1563	105.2188	0.8466	0.8471
19971231	6.000	103.7813	103.8438	0.8203	0.8208
19980630	5.125	100.0625	100.1250	0.7944	0.7950
19981231	5.125	99.6250	99.6875	0.7703	0.7708

Data Source: CRSP.

Table 2.8 The Price of a 15% Fixed Coupon Bond

Date	Cash Flow	Discount $Z(0,T)$	Discounted Cash Flow
19931231			
19941231	0.15	0.9642	0.1446
19951231	0.15	0.9193	0.1379
19961231	1.15	0.8745	1.0057
		Sum	1.2883
		Price $(\times 100)$	128.83

3. **Three-year floating rate bond**. From Section 2.5, we recall that the value of a floating rate bond is always equal to par at reset dates. Thus, we have $P_{FR}(3) = \$100$.

In conclusion, the value of the inverse floater is given by

$$\text{Price inverse floater} = P_z(0,3) + P_c(0,3) - P_{FR}(0,3) \quad (2.44)$$

$$= \$87.45 + \$128.83 - \$100 \quad (2.45)$$

$$= \$116.28 \quad (2.46)$$

2.8.4 Leveraged Inverse Floaters

Within Orange County's portfolio there were many different types of inverse floaters (e.g. different maturities and maximum interest rates). In addition, the portfolio contained some leveraged inverse floaters.[17] The main difference between these and the plain vanilla inverse

[17] See Mark Grinblatt and Sheridan Titman, Financial Markets and Corporate Strategy (2nd Edition), McGraw-Hill Primis, 2006, Chapter 23.

floaters discussed earlier is that the parity of floating rate to fixed rate is greater than one. For example, consider a 3-year leveraged inverse floater that pays a coupon of 25% minus two times the short-term interest rate. To price this security, we need to revise the steps we took to price inverse floaters.[18] The coupon is given by

$$c(t) = 25\% - 2 \times r_1(t-1) \tag{2.47}$$

What is a portfolio of bonds that pays this cash flow? A portfolio that is long a 25% fixed coupon bond and short *two* floating rate bonds achieves the coupon described in Equation 2.47. However, such a position at maturity entails that we receive \$100 from the long position and we must pay \$200 from the short position. In order to receive \$100 overall, we must also be long two zero coupon bonds. Thus, overall, we have

$$\text{Price leveraged inverse floater} = 2 \times P_z(0,3) + P_c(0,3) - 2 \times P_{FR}(0,3) \tag{2.48}$$

We already know from the previous section the prices $P_z(0,3) = \$87.45$ and $P_{FR}(0,3) = \$100$. The computation of the price of fixed-coupon bond with coupon rate equal to 25% yields a price of $P_c(0,3) = \$156.41$. Thus, we immediately find

$$\begin{aligned} \text{Price leveraged inverse floater} &= 2 \times \$87.45 + \$156.41 - 2 \times \$100 && (2.49)\\ &= \$131.32 && (2.50) \end{aligned}$$

This case study illustrates that we can readily apply the tools covered in this chapter to value more complex securities, such as inverse floaters and leveraged inverse floaters. In the next chapter we will follow up with this analysis and study the risk embedded in these securities. Finally, we note that dealing with real data and real markets often poses additional problems in the valuation and risk analysis of fixed income instruments: For instance, the computation of the discount curve $Z(0,t)$ requires the analysis of the data in Table 2.6, which is not straightforward. The next sections illustrates additional methodologies used in practice to deal with such large quantities of data.

2.9 APPENDIX: EXTRACTING THE DISCOUNT FACTORS $Z(0,T)$ FROM COUPON BONDS

The Orange County case study in the previous section makes it apparent that the bootstrap methodology discussed in Section 2.4.2 has limited applicability, and this for two reasons. First, for short-term maturities, there are too many bonds that mature on the same day to choose from. To perform the bootstrap methodology, we then must cherry pick the bonds that we deem have the highest liquidity (e.g., notes over bonds). Second, for longer maturities not all of the bonds may be available. In this case, some approximation is necessary. Sometimes it is possible to use the bonds that expire a few days earlier or later than the ones in the six-month cycle needed for the bootstrap. But often the gap across maturities may span longer periods, in which case the bootstrap methodology does not work well.

[18] We maintain the assumption that the coupon is always positive, that is that rates are always below 25%/2

2.9.1 Bootstrap Again

The iterative procedure described in the text is simple, but cumbersome. An easier way to obtain the same result is to use the matrix notation. Let $t = 0$, for convenience, so that T denotes both maturity date and time to maturity. Every coupon bond i is characterized by a series of cash flows and a maturity T^i. We can denote the total cash flow paid at time T_j as $c^i(T_j)$. In particular, denoting c^i the coupon rate of bond i, we have $c^i(T_j) = 100 \times c^i/2$ for $T_j < T^i$ and $c^i(T^i) = 100 \times (1 + c^i/2)$ and finally $c^i(T_j) = 0$ for $T_j > T^i$. We can put these cash flows in a row vector as follows:

$$\mathbf{C}^i = \left(c^i\left(T_1\right), c^i\left(T_2\right), ..., c^i\left(T_n\right)\right)$$

We can denote by $\mathbf{Z}\left(0\right)$ the vector of discount factors for various maturities T_i, that is

$$\mathbf{Z}\left(0\right) = \begin{pmatrix} Z\left(0,T_1\right) \\ Z\left(0,T_2\right) \\ \vdots \\ Z\left(0,T_n\right) \end{pmatrix}$$

The price of a coupon bond can be written using vector multiplication as:

$$P_c^i\left(0,T\right) = \mathbf{C}^i \times \mathbf{Z}\left(0\right)$$

We can denote the column vector of bond prices available at time 0 as

$$\mathbf{P}\left(0\right) = \begin{pmatrix} P_c\left(0,T_1\right) \\ P_c\left(0,T_2\right) \\ \vdots \\ P_c\left(0,T_n\right) \end{pmatrix}$$

We then obtain a system of n equations with n unknowns [the unknowns are the values of $Z\left(0,T_1\right), ..., Z\left(0,T_n\right)$]

$$\mathbf{P}\left(0\right) = \mathbf{C} \times \mathbf{Z}\left(0\right)$$

where $\mathbf{C}$ is the cash flow matrix:

$$\mathbf{C} = \begin{pmatrix} c^1\left(T_1\right) & c^1\left(T_2\right) & ... & c^1\left(T_n\right) \\ c^2\left(T_1\right) & c^2\left(T_2\right) & ... & c^2\left(T_n\right) \\ \vdots & & \ddots & \vdots \\ c^n\left(T_1\right) & c^n\left(T_2\right) & ... & c^n\left(T_n\right) \end{pmatrix}$$

Essentially, each row i of $\mathbf{C}$ corresponds to the cash flows of bond i for all the maturities T_1,...,T_n. In contrast, each column j describes all the cash flows that occur on that particular maturity T_j from the n bonds. The discount factors can then be obtained by inverting the cash flow matrix:

$$\mathbf{Z}\left(0\right) = \mathbf{C}^{-1} \times \mathbf{P}\left(0\right)$$

2.9.2 Regressions

As mentioned, we rarely have such nicely spaced data. Sometimes we in fact have too many maturities and sometimes we do not have enough maturities available to carry out the bootstrap procedure. The regression methodology deals with the case in which there are too many bonds compared to the number of maturities. This is typically the case when we consider maturities up to five years. For instance, in Table 2.6 there are 164 bonds with maturity of less than five years, but there are only 60 months in five years, implying that many months have multiple bonds maturing in them.

When we compute the cash flow matrix:

$$\mathbf{C} = \begin{pmatrix} c^1(T_1) & c^1(T_2) & ... & c^1(T_n) \\ c^2(T_1) & c^2(T_2) & ... & c^2(T_n) \\ \vdots & & \ddots & \vdots \\ c^N(T_1) & c^n(T_2) & ... & c^n(T_n) \end{pmatrix}$$

we end up with N rows ($N=$ number of bonds) and $n<N$ columns ($n=$ number of maturities). Since the solution to bootstrap involves inverting the matrix $\mathbf{C}$, we have a problem, as the matrix $\mathbf{C}$ must have an equal number of rows and columns to be inverted.

We can slightly change the bootstrap methodology to deal with this problem. For every bond $i=1,...,N$ let

$$P_c^i\left(0,T^i\right) = \mathbf{C}^i \times \mathbf{Z}(0) + \varepsilon^i \tag{2.51}$$

where ε^i is a random error term that captures any factor that generates the "mispricing." These factors include data staleness, lack of trading or liquidity. If we look at Equation 2.51, we see a regression equation of the type

$$y^i = \alpha + \sum_{j=1}^{n} \beta^j x^{ij} + \varepsilon^j$$

where the data are $y^i = P_c^i(0,T^i)$ and $x^{ij} = C_{ij}$, and the regressors are $\beta^j = Z(0,T_j)$. From basic Ordinary Least Squares (OLS) formulas, we then find

$$\mathbf{Z}(0) = \left(\mathbf{C}' \times \mathbf{C}\right)^{-1} \mathbf{C}' \times \mathbf{P}(0)$$

For this procedure to work, however, we must have more bonds than maturities, which does not occur for longer maturities. Curve fitting treats this latter problem.

2.9.3 Curve Fitting

Let's consider approaching the problem from a completely different angle. In particular, we can postulate a parametric functional form for the discount factor $Z(0,T)$ as a function of maturity T and use the current bond prices to *estimate* the parameters of this functional form. One popular model is the following:

2.9.3.1 The Nelson Siegel Model The Nelson Siegel model is perhaps the most famous model. The discount factor is posited to be given by

$$Z(0,T) = e^{-r(0,T)T} \tag{2.52}$$

where the continously compounded yield with maturity T is given by

$$r(0,T) = \theta_0 + (\theta_1 + \theta_2)\,\frac{1 - e^{-\frac{T}{\lambda}}}{\frac{T}{\lambda}} - \theta_2 e^{-\frac{T}{\lambda}} \tag{2.53}$$

where θ_0, θ_1, θ_2 and λ are parameters to be estimated from the current bond data.

The estimation proceeds as follows. For given parameter values $(\theta_0, \theta_1, \theta_2, \lambda)$, it is possible to compute the value of bond prices implied by the Nelson Siegel model. For each bond $i = 1, .., N$ with coupon c^i and cash flow payment dates maturity T_j^i, for $j = 1, .., n^i$, the Nelson Siegel model implies that the bond price should be

$$P_c^{i,\ NS\ model} = 100 \times \left(\frac{c^i}{2} \sum_{j=1}^{n_i} Z(0, T_j^i) + Z(0, T^i) \right) \tag{2.54}$$

For the same bond, we have the price quoted in the market, $P_c^{i,\ data}$ (note that this has to be the *invoice* price and not the *quoted* price). For each given set of parameters $(\theta_0, \theta_1, \theta_2, \lambda)$ we can compute the difference between model prices and data. Namely, we can compute

$$J(\theta_0, \theta_1, \theta_2, \lambda) = \sum_{i=1}^{N} \left(P_c^{i,\ NS\ model} - P_c^{i,\ data} \right)^2 \tag{2.55}$$

The Nelson Siegel model works perfectly if the model prices equal the data, i.e., if for every $i = 1, ..., N$ we have $P_c^{i,\ NS\ model} = P_c^{i,\ data}$. In this case, $J(\theta_0, \theta_1, \theta_2, \lambda) = 0$. The set of parameters $(\theta_0, \theta_1, \theta_2, \lambda)$ that achieves this objective would be the one to use to determine the discount factors $Z(0, T)$.

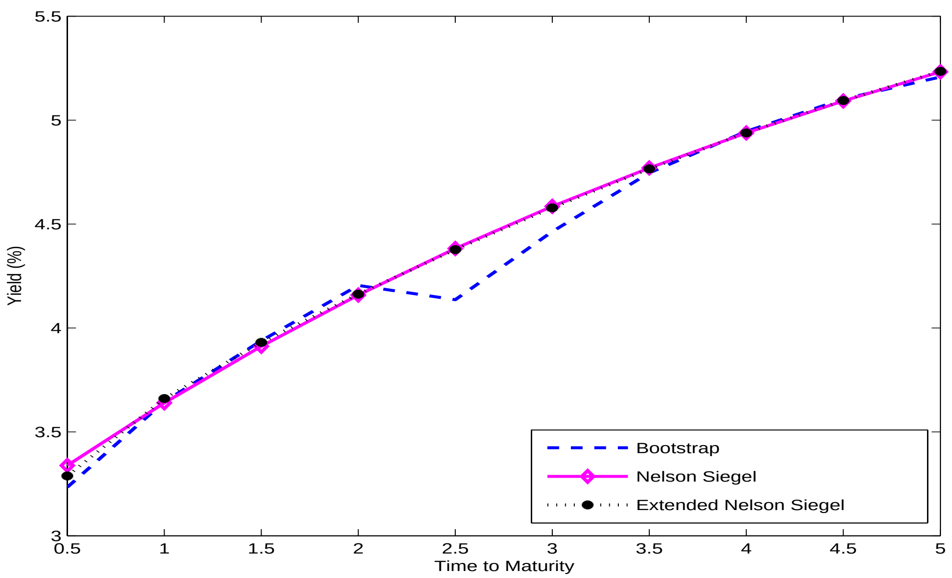
In general, however, it will not be possible to find parameter values that price all of the bonds exactly, because of staleness in the data, lack of liquidity, or lack of degrees of freedom in the Nelson Siegel model (we only have four parameters, after all). Therefore, the procedure is instead to find parameters $(\theta_0, \theta_1, \theta_2, \lambda)$ that *minimize* the quantity $J(\theta_0, \theta_1, \theta_2, \lambda)$ in Equation 2.55.

Figure 2.6 compares three methodologies of computing the term structure of interest rates: The bootstrap, the Nelson Siegel model, and the Extended Nelson Siegel model, further discussed below. The data are those contained in Table 2.6. The results of the bootstrap methodology are already contained in Table 2.7 in the form of discount function $Z(0, T)$. As it can be seen, the bootstrap method generates a yield curve that has a dip at maturity $T = 2.5$. It is not clear from the data why the dip in yield is present at that point: It could be a liquidity issue, or staleness, or simply an error in the database. The problem with bootstrap is that correcting for these sources of imprecision is hard.

The solid line in Figure 2.6 plots the fitted yield curve according to the Nelson Siegel model. The parameter estimates are $\theta_0 = 0.0754, \theta_1 = -0.0453, \theta_2 = -7.3182 \times 10{-}009$ and $\lambda = 3.2286$. The Nelson Siegel curve cuts through the bootstrapped curve smoothly. If the dip of the 2.5 year yield was a data error, it gets corrected in the minimization of errors. Indeed, note that we did not use only the ten bonds in Table 2.7 to fit the Nelson Siegel model, but the whole of 161 bonds with maturity less than five years in Table 2.6.

Can the Nelson Siegel model fit all of these data reasonably well? Figure 2.7 plots both the bond prices (stars) and the model prices (diamonds) for the various maturities: The model works well if the stars are close to the diamonds. The figure shows that indeed for most bonds this is the case, indicating that the model is doing quite well. The figure also shows that indeed at $T = 2.5$ there is a star that differs substantially from the diamond. This is in fact the bond that makes the bootstrap methodology fail at that maturity: The price seems too high compared to what the Nelson Siegel model – and in fact all of the other bonds around it – would imply. This fact suggests that either there is a trading opportunity available, or that that data point is an aberration and should be corrected. Unfortunately, such a correction is not easy if we use the bootstrap methodology.

Figure 2.6 The Term Structure of Interest Rates on December 31, 1993



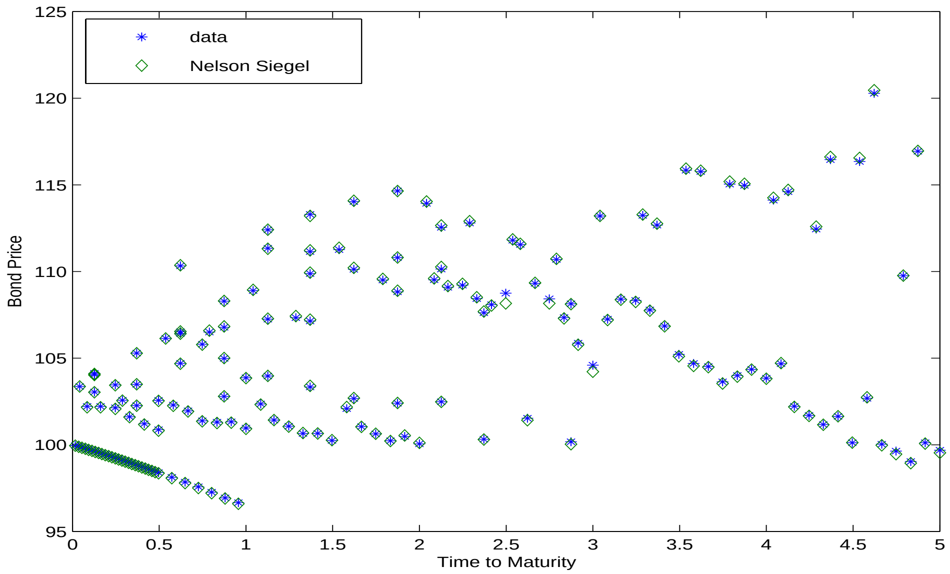
Source: Center for Research in Security Prices.

2.9.3.2 The Extended Nelson Siegel Model The Nelson Siegel model works well, but it lacks the flexibility to match term structures that are highly nonlinear. The economist Lars Svensson proposed an extension to the model, which is the one most widely adopted. In particular, we assume:

$$r(0,T) = \theta_0 + (\theta_1 + \theta_2)\frac{1 - e^{-\frac{T}{\lambda_1}}}{\frac{T}{\lambda_1}} - \theta_2 e^{-\frac{T}{\lambda_1}} + \theta_3\left(\frac{1 - e^{-\frac{T}{\lambda_2}}}{\frac{T}{\lambda_2}} - e^{-\frac{T}{\lambda_2}}\right) \quad (2.56)$$

where the parameters to estimate are 6: θ_i, $i = 0,..,3$ and λ_1 and λ_2. The procedure is otherwise the same as in the case of the Nelson Siegel model. Figure 2.6 shows the results of applying the extended Nelson Siegel model to the data in Table 2.6. The parameter estimates are $\theta_0 = 0.0687$, $\theta_1 = -0.0422$, $\theta_2 = -0.2399$, $\theta_3 = 0.2116$, $\lambda_1 = 0.9652$, and $\lambda_2 = 0.8825$. The outcome of the two procedures is almost the same. Indeed, the extended Nelson Siegel model has been put forward to capture severe non-linearities in the shape of the term structure of interest rates, a situation that did not occur in 1993.

Figure 2.7 The Fit of the Nelson Siegel Model



2.9.4 Curve Fitting with Splines

This is an extension of the curve fitting methodology described in Section 2.9.3, with a different specification of the discount factor $Z(t,T)$ as a function of maturity T. In essence, the idea is to assume that the discount function $Z(t,T)$ is given by a weighted average of basis functions $f_\ell(T)$, where the weights are chosen to best match the bond prices.

Specifically, the discount function is given by

$$Z(t,T) = 1 + \sum_{\ell=1}^{L} a_\ell f_\ell(T) \tag{2.57}$$

What are the functions $f_\ell(T)$? Many alternatives have been proposed.

1. **Simple polynomials:**

$$f_\ell(T) = T^\ell$$

This is the simplest case, where the discount function is the Lth-order polynomial

$$Z(t,T) = 1 + \sum_{\ell=1}^{L} a_\ell T^\ell$$

and the coefficients a_ℓ have to be estimated to minimize the distance between observed prices in the data $P_c^i(0,T_i)$ and the theoretical prices $\widehat{P}^i(0,T_i) = \sum_{j=1}^{n_i} c\left(T_j^i\right) \times Z\left(0, T_j^i\right)$, where T_j^i is the jth-s cash-flow date of bond i. The problem with polynomial functions is that they do not allow for a sufficient number of shapes, without going into a very high order polynomial. In this case, however, the discount function lacks the necessary stiffness to avoid being contaminated by small errors in data.

2. **Piecewise polynomial functions, or, spline**: Intuitively, a polynomial spline can be thought of as a number of separate polynomial functions, joined smoothly at a number of so called “joints”, “breaks,” or “knot” points. Using this method, each polynomial can be of low order and hence retain some stiffness, that is, a more stable curve. Cubic splines are the most used functions (so, third order), as they generate smooth forward curves. Of course, inside the family of splines there are many specifications, such as (to give some names):

   (a) Exponential cubic splines; and

   (b) B-splines.

   However, a number of other problems arise with these functions, the most important being the decision of *how many* knot points to include and, in addition, *where* to position them. We do not delve any more into this issue, as it is beyond the scope of this chapter, but relevant readings are available in the references to this chapter at the end of the book.

CHAPTER 3

BASICS OF INTEREST RATE RISK MANAGEMENT

Interest rates change substantially over time, and their variation poses large risks to financial institutions, portfolio managers, corporations, governments, and, in fact, households. Anyone who directly or indirectly either invest in fixed income securities or borrow money is subject to interest rate risks. This chapter discusses the basics of interest rate risk management. In particular, we discuss first how to measure risk for fixed income instruments, by introducing the notion of duration, value-at-risk and expected shortfall. Then, we also cover the basic techniques to mitigate financial risk, such as immunization and asset liability management.

3.1 THE VARIATION IN INTEREST RATES

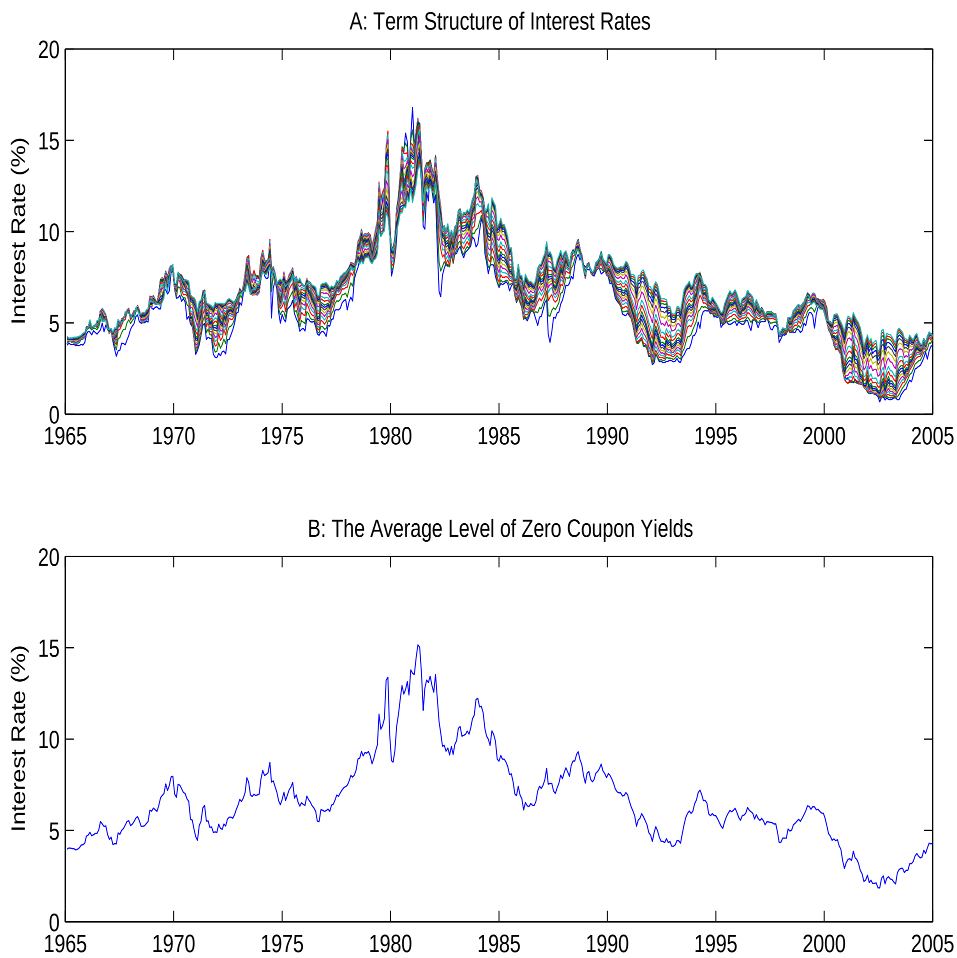
Interest rates change substantially over time. Panel A of Figure 3.1 plots the time series of yields from 1965 to 2005.[1] The various lines, all very close to each other, are the continuously compounded yields of zero coupon bonds for maturities from 1 month to 10 years. The most immediate fact that springs out from Panel A of this figure is that all yields move up and down roughly together. For instance, they were all relatively low in the 1960s,

[1]The spot rate curves are calculated by fitting the extended Nelson Siegel model to coupon bond data from CRSP (Monthly Treasuries) ©2009 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business.

Fixed Income Securities. By Pietro Veronesi

they were all relatively high in the late 1970s and early 1980s, and they were all relatively low in the late 1990s.

Figure 3.1 Zero Coupon Bond Yields and the Level of Interest Rates: 1965 - 2005



Data Source: CRSP.

Panel B of Figure 3.1 plots the simple average of yields graphed in Panel A. We may refer to this average generically as the level of interest rates.

Definition 3.1 *The* **level of interest rates** *is the average yield across maturities.*

As the level of interest rates changes over time, banks, bond portfolio managers and corporations worry about the impact that the swings in interest rates have on the value of their assets and liabilities. Two famous examples come to mind, namely, the savings and loan crisis in the 1980s and the bankruptcy of Orange County, California, in 1994.

3.1.1 The Savings and Loan Debacle

The savings and loan debacle in the 1980s is a standard example of what can go wrong when interest rates shift. A savings and loan earns a large part of its revenues from the difference between the long-term mortgages it provides to home owners and the short-term deposit rate it offers to depositors. When interest rates increased at the end of the 1970s, savings and loans were receiving their fixed coupons from mortgages contracted in the past, when rates were low, but suddenly they had to pay interest on deposits at the new higher deposit rates. Because depositors could choose where to put their money, banks were forced to offer high deposit rates, otherwise depositors would withdraw their deposits and invest in other securities, such as Treasuries. A withdrawal of funds is the worst nightmare for a bank, as depositors' money is not in the bank any longer: It has been loaned to others. The spread between the rate earned on assets and the (higher) rate paid on liabilities quickly put many savings and loans out of business.

3.1.2 The Bankruptcy of Orange County

In 1994 Orange County, California, lost \$1.6 billion when the interest rate unexpectedly increased from 3% to 5.7% over the course of the year.[2] The substantial loss from the total asset pool of \$7.5 billion forced Orange County to declare bankruptcy. Through the use of a mix of structure notes and leverage, Orange County's portfolio stood to make subtantial above market returns were the interest rate not to increase in the near future. But interest rates did increase, and the fund went down. This famous case highlights yet another example of the large losses that interest rate movements may bring about in portfolios that are "too sensitive" to interest rates.

3.2 DURATION

The examples above calls for (a) a systematic methodology to assess the riskiness of a bond portfolio to movements in interest rates; and (b) a methodology to effectively manage such risk. We tackle the former problem in this section, and the latter in the next.

Definition 3.2 *The* **duration** *of a security with price P is the (negative of the) percent sensitivity of the price P to a small* parallel *shift in the level of interest rates. That is, let $r(t,T)$ be the continuously compounded term structure of interest rates at time t. Consider a uniform shift of size dr across rates that brings rates to $\overline{r}(t,T)$, given by*

$$r(t,T) \longrightarrow \overline{r}(t,T) = r(t,T) + dr$$

Let the price of the security move by dP as a consequence of the shift:

$$P \longrightarrow \overline{P} = P + dP$$

[2]See the case study ERISK: Orange County, downloaded from the web site http://www.erisk.com/Learning/CaseStudies/OrangeCounty.asp.

The duration of the asset is then defined as[3]

$$\textit{Duration} = D_P = -\frac{1}{P}\frac{dP}{dr} \tag{3.1}$$

The shift dr is a small uniform change across maturities, such as, for instance, 1 basis point: $dr = .01\%$. The notion of duration then measures the impact that such a uniform change on the yield curve has on the price of the security P. This can be seen by reorganizing Equation 3.1 as follows:

Fact 3.1 *Given a duration D_P of a security with price P, a uniform change in the level of interest rates brings about a change in value of*

$$\textit{Change in portfolio value} = dP = -D_P \times P \times dr \tag{3.2}$$

■ EXAMPLE 3.1

A \$100 million bond portfolio has duration equal 10, $D_P = 10$. This implies that one basis point increase in the level of interest rates $dr = .01\%$ generates a swing in portfolio value of

$$\begin{aligned} \text{Change in portfolio value} = dP &= -10 \times \$100 \text{ million} \times .01/100 \\ &= -\$100,000 \end{aligned}$$

That is, the portfolio manager stands to lose \$100,000 for every basis point increase in the term structure.

How can we compute the duration of a security? Before we can answer this important question, however, we need to recall the following two concepts from calculus. To simplify our analysis, we will only consider continuously compounded interest rates, as in the definition above. Below, we also review the more traditional notion of duration that uses semi-annually compounded yield to maturity in its definition.

Definition 3.3 *Let A and a be two constants and x be a variable. Let $F(x) = A \times e^{ax}$ be a function of x. Then, the first* **derivative** *of F with respect to x, denoted by dF/dx, is given by*

$$\textit{Derivative of } F(x) \textit{ with respect to } x = \frac{dF}{dx} = A \times a \times e^{ax} = a \times F(x) \tag{3.3}$$

An example of the function $F(x)$ is the zero coupon bond formula studied in Chapter 2

$$P_z(t,T) = 100 \times Z(t,T) = 100 \times e^{-r(T-t)}.$$

[3]The duration definition in Equation 3.1 is often referred to as the "modified duration," to differentiate it from the Macaulay definition of duration, discussed below. In this book, we will rarely use the Macaulay duration, and therefore we reserve the term duration for modified duration.

In this case the constant A is the notional 100, the constant a equals the time to maturity $T-t$, and the variable x equals the continuously compounded interest rate r. The notion of the first derivative of $P_z(t,T)$ with respect to r then gives the sensitivity of the zero coupon bond to the interest rate r.

Fact 3.2 *Let $P_z(r,t,T)$ be the price of a zero coupon bond at time t with maturity T and continuously compounded interest rate r. The first derivative of $P_z(r,t,T)$ with respect to r is*

$$\begin{aligned}\frac{d\,P_z}{d\,r} &= 100\times\left[\frac{d\left(e^{-r(T-t)}\right)}{dr}\right]\\ &= 100\times\left[-(T-t)\times e^{-r(T-t)}\right]\\ &= -(T-t)\times P_z(r,t,T) \qquad (3.4)\end{aligned}$$

To emphasize that the zero coupon bond price depends on the current interest rate r, in this section we denote it by $P_z(r,t,T)$, that is, we add r as one of the arguments in $P_z(t,T)$. Visually, the first derivative represents the *slope* of the curve $P_z(r,t,T)$, plotted against r, at the current interest rate level. More specifically, Figure 3.2 plots the price of a 20-year zero coupon bond for various values of r, ranging from 0 to 15%. In the plot $T-t=20$, as the zero coupon bond has 20 years to maturity. Suppose today the interest rate is $r=6\%$. The straight dotted line in the Figure is the *tangent* to the curve $P_z(r,t,T)$ at the point $r=6\%$. The slope of this tangent is the first derivative of $P_z(r,t,T)$ with respect to r, dP_z/dr.

3.2.1 Duration of a Zero Coupon Bond

We are now in the position of computing the duration of a zero coupon bond. The only thing we have to remember is Definition 3.2, and the rule of the first derivative in Definition 3.3 when applied to a zero coupon bond (Equation 3.4). It is instructive to go through the steps to compute the duration of a zero coupon bond, $D_{z,T}$, where the notation "z" reminds us that this calculation is done for a zero coupon bond.

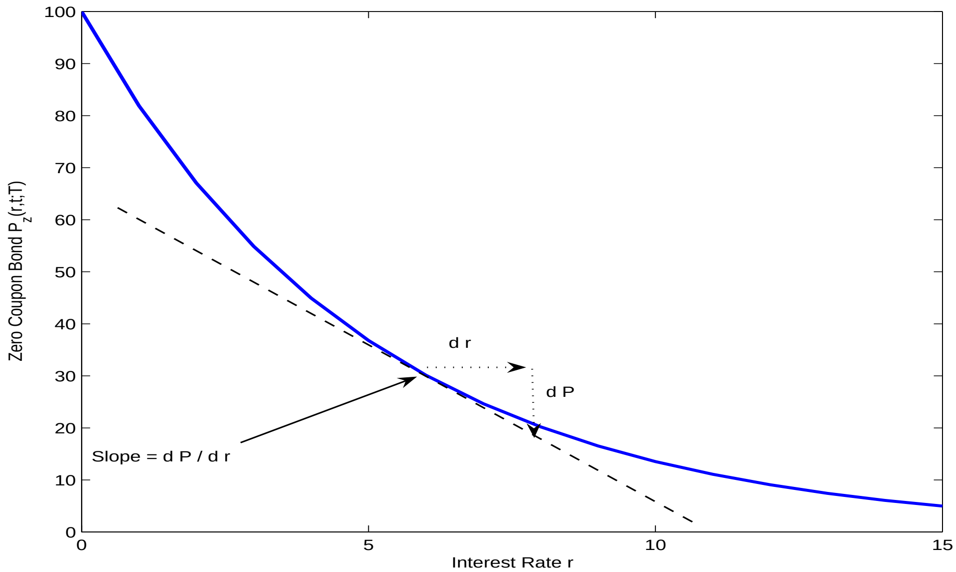
$$\begin{aligned}D_{z,T} &= -\frac{1}{P_z(r,t,T)}\left[\frac{dP_z(r,t,T)}{dr}\right] \qquad (3.5)\\ &= -\frac{1}{P_z(r,t,T)}\times[-(T-t)\times P_z(r,t,T)]\\ &= T-t \qquad (3.6)\end{aligned}$$

The duration of a zero coupon bond is given by its time to maturity $T-t$. This makes it very simple to compute, indeed.

■ EXAMPLE 3.2

A portfolio manager has \$100 million invested in 5-year STRIPS. The duration of this portfolio is then 5, implying that a one basis point increase in interest rates decreases

Figure 3.2 First Derivative of a Zero Coupon Bond with Respect to Interest Rate r



the value of the portfolio approximately by

$$dP \approx -D_P \times P \times dr = -5 \times \$100 \text{ million} \times .01\% = -\$50,000$$

3.2.2 Duration of a Portfolio

What is the duration of a portfolio of securities? Consider a portfolio made up of N_1 units of security 1, and N_2 units of security 2. Let P_1 and P_2 be the prices of these two securities, respectively. The value of the portfolio is then

$$W = N_1 \times P_1 + N_2 \times P_2$$

Let D_1 and D_2 be the duration of security 1 and security 2, respectively. By definition,

$$D_i = -\frac{1}{P_i}\frac{d\,P_i}{d\,r}$$

How can we determine the duration of the portfolio? We can apply the definition of duration in Definition 3.2 and reorganize the expressions:

$$\begin{aligned}\text{Duration of portfolio} &= D_W = -\frac{1}{W}\frac{d\,W}{d\,r} \quad (3.7)\\ &= -\frac{1}{W}\frac{d(N_1 \times P_1 + N_2 \times P_2)}{d\,r}\\ &= -\frac{1}{W}\left[N_1 \times \frac{dP_1}{d\,r} + N_2 \times \frac{dP_2}{d\,r}\right]\end{aligned}$$

$$\begin{aligned} &= \frac{1}{W}\left[N_1 \times P_1 \times \left(-\frac{1}{P_1}\frac{dP_1}{d\,r}\right) + N_2 \times P_2 \times \left(-\frac{1}{P_2}\frac{dP_2}{d\,r}\right)\right] \\ &= \frac{N_1 \times P_1}{W} D_1 + \frac{N_2 \times P_2}{W} D_2 \\ &= w_1 D_1 + w_2 D_2 \end{aligned} \tag{3.8}$$

where

$$w_1 = \frac{N_1 \times P_1}{W} \quad \text{and} \quad w_2 = \frac{N_2 \times P_2}{W} \tag{3.9}$$

The expression in Equation 3.8 shows that the duration of a portfolio is a weighted average of the durations of the assets, where the weights correspond to the percentage of the portfolio invested in the given security.

■ **EXAMPLE 3.3**

A bond portfolio manager has \$100 million invested in 5-year STRIPS and \$200 million invested in 10-year STRIPS. What is the impact of a one basis point parallel shift of the term structure on the value of the portfolio?

We can answer this question by computing the duration of the portfolio: The 5-year and 10-year strips have duration of 5 and 10, respectively. The total portfolio value is \$300 million. Thus, the duration of the portfolio is

$$\text{Duration of portfolio} = \frac{100}{300} \times 5 + \frac{200}{300} \times 10 = 8.3$$

Therefore, a one basis point increase in interest rates generates a portfolio loss of

$$\text{Loss in portfolio value} = \$300 \text{ million} \times 8.3 \times 0.01\% = \$249,000$$

Generalizing the formula in Equation 3.8 to n securities, we obtain:

Fact 3.3 *The duration of portfolio of n securities is given by*

$$D_W = \sum_{i=1}^{n} w_i D_i \tag{3.10}$$

where w_i is the fraction of the portfolio invested in security i, and D_i is the duration of security i.

3.2.3 Duration of a Coupon Bond

We can apply the result in Fact 3.3 to compute the duration of a coupon bond. As discussed in Section 2.4, a coupon bond with coupon rate c and n future coupon payments can be considered a portfolio of zero coupon bonds, in which $c/2$ is invested in the first $n-1$ zeros, and $1 + c/2$ in the $n-$th zero:

$$P_c(0, T_n) = \sum_{i=1}^{n-1} \frac{c}{2} \times P_z(0, T_i) + \left(1 + \frac{c}{2}\right) P_z(0, T_n) \tag{3.11}$$

The duration of a coupon bond can then be computed by using Equation 3.10. Define the weights

$$w_i = \frac{c/2 \times P_z(0,T_i)}{P_c(0,T_n)} \quad \text{for } i = 1,..,n-1$$

$$w_n = \frac{(1+c/2) \times P_z(0,T_n)}{P_c(0,T_n)}$$

Then, the duration of the coupon bond is

$$D_c = \sum_{i=1}^{n} w_i D_{z,T_i} \tag{3.12}$$

$$= \sum_{i=1}^{n} w_i T_i \tag{3.13}$$

That is, the duration of a coupon bond is a weighted average of coupon payment times T_i.

EXAMPLE 3.4

Consider a 10-year, 6% coupon bond. Given a discount curve $Z(0,T)$, we can compute its duration by following the calculations in Table 3.1. In this table, the second and third columns present the payment times and the payment amounts. The discount factor curve $Z(0,T)$ is in the fourth column. In the fifth column we compute the discounted cash flows, the sum of which give the price $P_c(0,T) = \$107.795$ at the bottom of the table. The weights in column 6 equal the discount cash flows in column 5 divided by the price. Finally, the last column reports the weighted payment times, $\omega_i \times T_i$. The duration is given by the sum of these weighted payment times, reported at the bottom of the table: $D_c = 7.762$. Different from zero coupon bonds, the duration of the coupon bond is shorter than its maturity.

3.2.4 Duration and Average Time of Cash Flow Payments

While we have derived the formula for duration in Equation 3.13 from the definition of duration as the percentage sensitivity of a security to changes in interest rates (see Definition 3.2), some confusion sometimes arise about the notion of duration because sometimes people *define* duration as the average time of payments, as in Equation 3.13 (see also Section 3.2.6). These two interpretations are equivalent for fixed rate bonds, that is, bonds that pay fixed coupons: A zero coupon bond with maturity of 5 years has duration equal to 5. This is both the average time of payments (there is only one), and also the percentage loss in value from an increase in interest rates. A similar situation exists for coupon bonds. However, for many securities that do not have fixed payments, the equivalence is broken. The following provides a simple example:

EXAMPLE 3.5

Table 3.1 Duration of Coupon Bond, Coupon = 6%

Period i	Payment Time T_i	Cash Flow CF	Discount $Z(0,T_i)$	Discounted Cash Flow CF $\times Z(0,T_0)$	Weight w_i	Weight $\times T_i$ $w_i \times T_i$
1	0.5	3	0.976	2.927	0.027	0.014
2	1.0	3	0.952	2.855	0.026	0.026
3	1.5	3	0.929	2.786	0.026	0.039
4	2.0	3	0.906	2.718	0.025	0.050
5	2.5	3	0.884	2.652	0.025	0.061
6	3.0	3	0.862	2.587	0.024	0.072
7	3.5	3	0.841	2.524	0.023	0.082
8	4.0	3	0.821	2.462	0.023	0.091
9	4.5	3	0.801	2.402	0.022	0.100
10	5.0	3	0.781	2.344	0.022	0.109
11	5.5	3	0.762	2.286	0.021	0.117
12	6.0	3	0.744	2.231	0.021	0.124
13	6.5	3	0.725	2.176	0.020	0.131
14	7.0	3	0.708	2.123	0.020	0.138
15	7.5	3	0.690	2.071	0.019	0.144
16	8.0	3	0.674	2.021	0.019	0.150
17	8.5	3	0.657	1.972	0.018	0.155
18	9.0	3	0.641	1.923	0.018	0.161
19	9.5	3	0.626	1.877	0.017	0.165
20	10.0	103	0.610	62.858	0.583	5.831
			Price	107.795	Duration	7.762

Recall that in Section 2.5 of Chapter 2 we considered the price of a floating rate bond. In particular, Fact 2.12 shows that if T_i denotes the last reset date, T_{i+1} denotes the next reset date, t is the current time, with $T_i < t < T_{i+1}$, then the price of a floating rate bond with maturity T and semi-annual payments is given by (see Equation 2.40)

$$P_{FR}(t,T) = Z(t,T_{i+1}) \times 100 \times [1 + r_2(T_i)/2] \tag{3.14}$$

where $r_2(T_i)$ is the reference rate that is determined at the last reset date. The duration of the floating rate bond is then

$$\begin{aligned}
\text{Duration of floating rate bond at } t \quad &= \quad D_{FR} = -\frac{1}{P_{FR}(t,T)}\frac{d\,P_{FR}}{d\,r} &(3.15)\\
&= \quad -\frac{1}{P_{FR}(t,T)}\left[\frac{d\,Z(t,T_{i+1})}{d\,r}\right] \times 100 \times \left[1+\frac{r_2(T_i)}{2}\right]\\
&= \quad -\frac{1}{P_{FR}(t,T)}\left[-(T_{i+1}-t)\right] \times Z(t,T_{i+1}) \times 100 \times \left[1+\frac{r_2(T_i)}{2}\right]\\
&= \quad T_i - t &(3.16)
\end{aligned}$$

where the last equality stems from using again Equation 3.14. Equation 3.16 shows that the duration of a floating rate bond is simply equal to the time left to the *next* coupon payment date $T_{i+1} - t$. In particular, if today is coupon date (but the coupon has not been paid yet), the duration is zero.

This example shows that even if the average time of future cash flows can be relatively long – a floating coupon bond with 10 years to maturity, for instance, has an average time of future payments of several years – the duration could be very small. Conversely, we will see securities for which the duration is actually longer than their maturity, or securities for which the duration is negative. Given that in modern times the notion of duration is mainly used for risk management purposes, and in particular to compute the sensitivity of a security to parallel shifts in the term structure, we must be careful in interpreting duration as an average time of future payments, as this interpretation *only* holds for securities with *fixed* cash flows.

3.2.5 Properties of Duration

It is important to realize that the duration of a coupon bond depends crucially on the level of the coupon rate. As the coupon rate increases, the duration is lower. The first three columns of Table 3.2 show this effect for the case in Example 3.4. What is the intuition? There are two ways to see this result intuitively:

1. *Lower Average Time of Cash Flow Payments:* The higher the coupon, the larger are the intermediate coupons relative to the last one (in 10 years). Thus, the average time of coupon payments gets closer to today.

2. *Lower Sensitivity to Interest Rates:* The higher the coupon rate, the larger are cash flows in the near future compared to the long-term future. Cash flows that arrive sooner rather than later are less sensitive to changes in interest rates (for instance, a cash flow arriving tomorrow has no sensitivity to interest rates). Thus, an increase in coupon rate implies an overall lower sensitivity to changes in discount rates.

Table 3.2 Duration versus Coupon Rate and Interest Rate

Coupon c	Price P_c	Duration D	Interest Rate r_2	Price P_c	Duration D
0	61.03	10.00	1%	147.47	8.13
2%	76.62	8.95	3%	125.75	7.95
4%	92.21	8.26	5%	107.79	7.76
6%	107.79	7.76	7%	92.89	7.56
8%	123.38	7.39	9%	80.49	7.35
10%	138.97	7.11	11%	70.12	7.12
12%	154.56	6.88	13%	61.44	6.90

For the same reason, the duration of a coupon bond decreases with the general level of interest rates, as we see for the last three columns of Table 3.2. In this case, the coupon rate is kept at 6%, but the semi-annual interest rate r_2 – constant across maturities – increases from 1% to 13%. Note that the duration drops from 8.13 to 6.90. Once again, a higher interest rate (across maturities) implies that short-term cash flows have a relatively higher weight in the value of the bond, and thus a lower sensitivity to changes in interest rates.

3.2.6 Traditional Definitions of Duration

We defined the duration as

$$D = -\frac{1}{P}\frac{dP}{dr} \tag{3.17}$$

where r is the continuously compounded interest rate. This definition of duration is simple to apply in order to compute the duration of interest rate securities, from zero coupon bonds to portfolios of securities. For instance, Equation 3.17 shows that the duration of a *fixed* coupon bond equals the average time of payment times, which is a relatively simple formula to determine the sensitivity of a coupon bond to parallel shifts in the yield curve.

Traditionally, however, the duration is not defined against the continuously compounded interest rate but rather against the semi-annually compounded yield to maturity. In this case, the definition of the modified duration as the (negative of the) sensitivity of prices to changes in interest rates (Equation 3.17) does not correspond exactly to the simple formulas derived earlier, and a small adjustment is needed. In particular, consider a security with yield to maturity y (see Section 2.4.3 in Chapter 2). Recall that by definition of yield to maturity, the price of the coupon bond on a coupon date can be written as

$$P_c(0,T) = \sum_{j=1}^{n} \frac{c/2 \times 100}{\left(1+\frac{y}{2}\right)^{2\times T_j}} + \frac{100}{\left(1+\frac{y}{2}\right)^{2\times T_n}} \tag{3.18}$$

A little algebra shows that the modified duration (MD) of this coupon bond, when defined against the yield to maturity y, is given by

$$MD = -\frac{1}{P}\frac{dP}{dy} = \frac{1}{\left(1+\frac{y}{2}\right)} \sum_{j=1}^{n} w_j \times T_j \tag{3.19}$$

where

$$w_j = \frac{1}{P_c(0,T)}\left(\frac{c/2 \times 100}{\left(1+\frac{y}{2}\right)^{2\times T_j}}\right),\ w_n = \frac{1}{P_c(0,T)}\left(\frac{100 \times (c/2+1)}{\left(1+\frac{y}{2}\right)^{2\times T_n}}\right)$$

In other words, when we use the semi-annual compounded yield to maturity y to define the modified duration, then the modified duration of a fixed rate bond can be computed as the weighted average of cash flow maturities times an adjustment given by $(1/(1+\frac{y}{2})$. The weighted average of cash flow maturities in Equation 3.19 is called the **Macaulay duration**

$$D^{Mc} = \sum_{j=1}^{n} w_j \times T_j \tag{3.20}$$

We will rarely use the variation in the semi-annually compounded yield to maturity for risk management purposes, and rather use the variation in the continuously compounded spot curve. Not only does this choice allow for simpler formulas, as we showed in the previous sections, but it also implies that the durations of different assets are defined against the variation of the same interest rates, namely, the spot rates. Instead, a definition in terms of yield to maturity hinges on the notion of yield to maturity itself, which, as discussed in Chapter 2 (Section 2.4.3) has some issues, such as the fact that it is bond specific, it depends on the coupon rate, and so on. In addition, for several fixed income securities the notion of yield to maturity is not well defined, because they may have floating rate coupons or embedded options. The notion of a parallel shift in the spot curve is always well defined for any interest rate security, and whenever an analytical formula is not available, we can always rely on computers to obtain an approximate quantity, called effective duration. We apply such a strategy for instance in Chapter 8 in the context of mortgage backed securities.

3.2.7 The Duration of Zero Investment Portfolios: Dollar Duration

The definition of duration in Equation 3.1 implicitly implies that the security, or the portfolio, has nonzero value. However, in many interesting cases involving no arbitrage strategies, the security or the portfolio may have a value of exactly zero. In this case, we resort to the dollar duration:

Definition 3.4 *The* **dollar duration** $D^{\$}$ *of a security* P *is defined by*

$$\textit{Dollar duration} = D_P^{\$} = -\frac{dP}{dr} \tag{3.21}$$

That is, the dollar duration corresponds to the (negative of the) sensitivity of the price P to changes in the level of interest rate r. Since dP is the change in the price of the security, in dollars, the name dollar duration follows. Combining Equations 3.1 and 3.21 we obtain:

Fact 3.4 *For a nonzero valued security or portfolio with price* P*, the relation between duration and dollar duration is*

$$D_P^{\$} = P \times D_P \tag{3.22}$$

In this case, the relation between the dollar duration of the portfolio and the dollar duration of its individual components is given by the sum of the dollar durations:

Fact 3.5 *The dollar duration of portfolio of n securities, denoted by $D_W^\$$ is given by*

$$D_W^\$ = \sum_{i=1}^{n} N_i D_i^\$ \tag{3.23}$$

where N_i is the number of units of security i in the portfolio, and $D_i^\$$ is the dollar duration of security i.

■ **EXAMPLE 3.6**

Dollar Duration of a Long-Short Strategy

Let the term structure of interest rates be flat at 4% (semi-annually compounded). Consider an arbitrageur who is contemplating going long a 4% coupon bond by borrowing at the current floating rate. To keep the analysis simple, suppose the arbitrageur can enter into term repos with maturity of six months and that the haircut is zero. Because the term structure of interest rate is flat, a 4% coupon bond would be valued at par ($100), which is the amount the arbitrageur needs to borrow. The portfolio has value of zero at time $t = 0$ when the trade is set up. However, the trade is risky, because if interest rates move up, then the arbitrageur will suffer a decrease in value in the long position that is not paralleled by an equivalent decrease in value in the short (borrowing) position.

More specifically, borrowing at the 6-month term repo is essentially equivalent to shorting a 6-month floating rate bond. Thus, the long-short portfolio can be written as

$$W = P_c(0,T) - P_{FR}(0,T) = 0$$

Let the duration of the fixed rate bond be 8.34. The duration of the floating rate bond is 6 months, as discussed in Example 3.5. Thus, using Equation 3.22, we find

$$\begin{aligned}
\text{Dollar duration of } P_c(0,T) &= \$100 \times \text{Duration of fixed rate bond} \\
&= \$100 \times 8.34 = \$834 \\
\text{Dollar duration of } P_{FR}(0,T) &= \$100 \times \text{Duration of floating rate bond} \\
&= \$100 \times 0.5 = \$50
\end{aligned}$$

Thus, the dollar duration of the long-short portfolio is

$$\text{Dollar duration of long-short portfolio} = \$834 - \$50 = \$784 \tag{3.24}$$

Again using Equation 3.21, we have that one basis point increase in interest rate dr generates change in the long-short portfolio:

$$\text{Change in portfolio value} = d\,W = -D_W^\$ \times d\,r = -\$784 \times .01/100 = -.0784 \tag{3.25}$$

That is, the long-short portfolio with trade size of $1 million, for instance, stands to lose $78,400 for every basis point increase in the level of interest rates.

The dollar losses due to a basis point increase in the level of interest rates, as computed in Equation 3.25 is a common measure of interest rate risk. Traders refer to it as the "price value of a basis point", or PVBP, or PV01:

Definition 3.5 *The* **price value of a basis point** *PV01 of a security with price P is defined as*

$$\textit{Price value of a basis point} = PV01(or PVBP) = -D_P^{\$} \times d\, r \qquad (3.26)$$

3.2.8 Duration and Value-at-Risk

Value-at-Risk (VaR) is a risk measure that attempts to quantify the amount of risk in a portfolio. In brief, VaR answers the following question: With 95% probability, what is the maximum portfolio loss that we can expect within a given horizon, such as a day, a week or a month? Methodologies for the computation of VaR are many and range from highly complex to fairly simple. In this section we discuss two methodologies that are based on the concept of duration: The historical distribution approach and the normal distribution approach.

Definition 3.6

Let α be a percentile (e.g. 5%) and T a given horizon. The $(100-\alpha)\%$ T **Value-at-Risk** *of a portfolio P is the maximum loss the portfolio can suffer over the horizon T with $\alpha\%$ probability. In formulas, let $L_T = -(P_T - P_0)$ denote the loss of a portfolio over the horizon T (a profit if negative). The VaR is that number such that:*

$$Prob\,(L_T > VaR) = \alpha\% \qquad (3.27)$$

For instance, a \$100 million bond portfolio may have a 95%, 1-month VaR of \$3 million. This VaR measure implies that there is only 5% probability that the portfolio losses will be higher than \$3 million over the next month.

The VaR measure is based on the volatility of the underlying assets in the portfolio. For bond portfolios, the volatility is determined by movements in the interest rates. In fact, through duration, we can estimate the sensitive of a portfolio to fluctuations in the interest rate. Recall Equation 3.2:

$$dP = -D_P \times P \times dr \qquad (3.28)$$

Given the value of the portfolio P and its duration D_P, we can transform the probability distribution of interest rate changes dr into the probability distribution of portfolio changes dP, and from the latter, we can compute the portfolio potential losses. The 95% VaR corresponds to the 5% worst case of distribution of dP. A simple example is given by the case in which dr has a normal distribution:

Fact 3.6 *Let dr have a normal distribution with mean μ and standard deviation σ. Then Equation 3.28 implies that dP has a normal distribution with mean and standard deviation given by:*

$$\mu_P = -D_P \times P \times \mu \quad \textit{and} \quad \sigma_P = D_P \times P \times \sigma. \qquad (3.29)$$

That is:

$$dr \sim N(\mu, \sigma^2) \quad \Longrightarrow \quad dP \sim N(\mu_P, \sigma_P^2) \tag{3.30}$$

The 95% VaR is then given by

$$\textit{95\% VaR} = -(\mu_P - 1.645 \times \sigma_P) \tag{3.31}$$

where -1.645 *corresponds to the* 5*-th percentile of the standard normal distribution, that is, if* $x \sim \mathcal{N}(0,1)$ *then* $Prob(x < -1.645) = 5\%$*. The 99% VaR is computed as in Equation 3.31 except that the number "1.645" is substituted by "2.326."*

This result of course relies on Equation 3.28, which is only an approximation. If dr is not normal, Equation 3.31 does not hold. The next example illustrates one popular approach to dealing with this latter case.

EXAMPLE 3.7

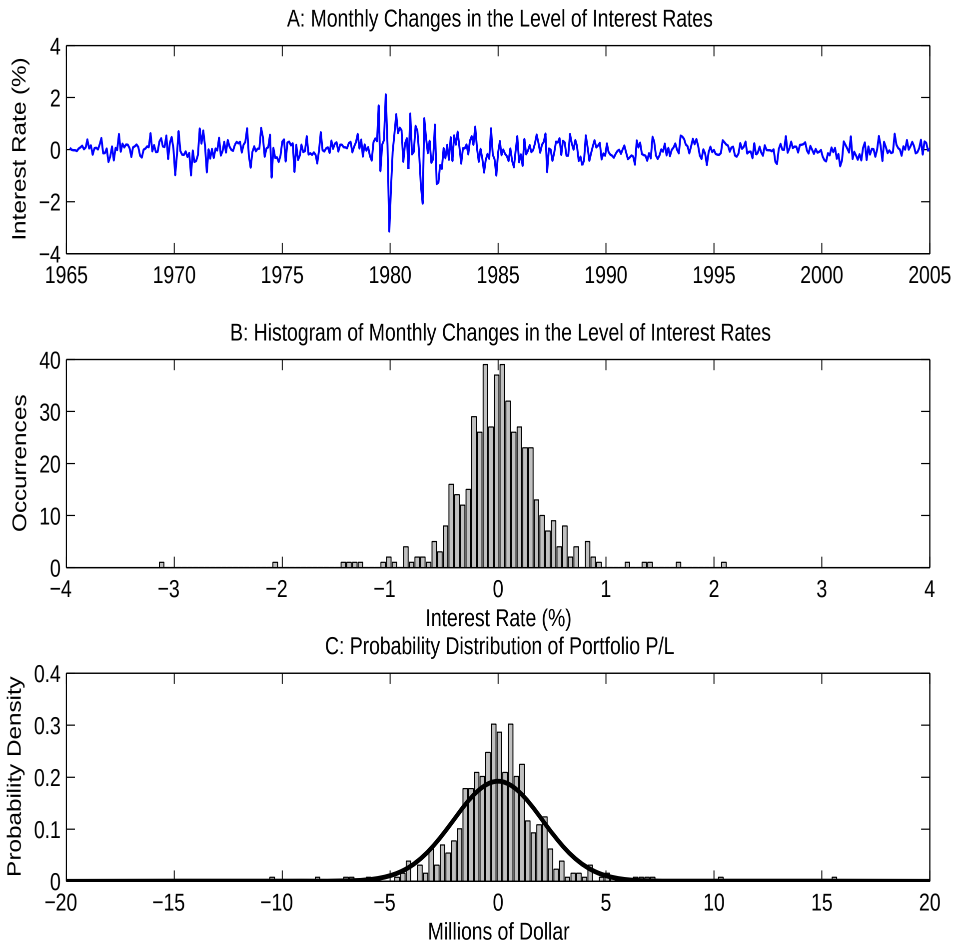
A portfolio manager has \$100 million invested in a bond portfolio with duration $D_P = 5$. What is the 95% one-month Value-at-Risk of the portfolio?

1. **Historical Distribution Approach.** We can use the past changes in the level of interest rates dr as a basis to evaluate the potential changes in a portfolio value dP. Panel B of Figure 3.1 shows the historical observations of the level of interest rates up to 2005. Panel A of Figure 3.3 shows the monthly changes in the level of interest rates, while Panel B makes a histogram of these variations. As we can see large increases and decreases are not very likely, but they do occur occasionally. We can now multiply each of these changes dr observed in the plot by $-D_P \times P = -5 \times 100$ million to obtain the variation in dP. Panel C of Figure 3.3 plots the histogram of the changes in the portfolio i.e., the portfolio profits and losses (P&L).[4] Given this distribution, we can compute the maximum loss that can occur with 95% probability. We can start from the left-hand side of the distribution, and move rightward until we count 5% of the observations. That number is the 95% monthly VaR computed using the historical distribution approach. In this case, we find it equal to \$3 million. That is, there is only 5% probability that the portfolio losses will be higher than \$3 million.

2. **Normal Distribution Approach.** From Fact 3.6, a normal distribution assumptin on dr translates into a normal distribution on dP. Using the data plotted in Panel A of Figure 3.3, we find that the monthly change in interest rate has mean $\mu = 6.5197 \times 10 - 006$ and stadard deviation $\sigma = .4153\%$. Therefore, $\mu_P = -5 \times 100 \times \mu = -.0033$ and $\sigma_P = 5 \times 100 \times \sigma = 2.0767$. The standard normal distribution is plotted along with the (renormalized) histogram in Panel C of Figure 3.3. The 95% VaR is then equal to 95% VaR $= -(\mu_P - 1.645 \times \sigma_P) = \$3.4194$ million.

[4]We renormalized the histogram to make it comparable with the normal distribution case, discussed in the next point.

Figure 3.3 Changes in the Level of Interest Rates: 1965 - 2005



Data Source: CRSP.

3.2.8.1 Warnings It is worth emphasizing immediately a few problems with the Value-at-Risk measure of risk, as well as some potential pitfalls:

1. VaR is a statistical measure of risk, and as with any other statistical measure, it depends on distributional assumptions and the sample used for the calculation. The difference can be large. For instance, in Example 3.7 the VaR varies depending on whether we use the normal distribution approach or the historical distribution approach.

2. The duration approximation in Equation 3.28 is appropriate for *small* parallel changes in the level of interest rates. However, by definition, VaR is concerned with *large* changes. Therefore, the duration approximation method is internally incosistent.

The problem turns out to be especially severe for portfolios that include derivative securities, either implicitly or explicitly. We will return to this issue in later chapters.

3. The VaR measures the maximum loss with 95% probability. However, it does not say anything about how large the losses could be if they do occur. The tails of the probability distribution matter for risk. For instance, in Example 3.7 the 99% VaR using the historical distribution approach is $5.52 million, while this figure is only $4.83 million using the normal distribution assumption. The tails of the normal distribution are thin, in the sense that they give an extremely low probability to large events, which instead in reality occur with some frequency.

4. The VaR formula used in Equation 3.31 includes the *expected* change in the portfolio $\mu_P = -D_P \times P \times E[dr]$. The computation on the expected change $E[dr]$ is typically very imprecise, and standard errors are large. Such errors can generate a large error in the VaR computation. For this reason, it is often more accurate to consider only the *unexpected VaR*, that is, consider only the 95% loss compared to the expected P&L μ_P. Practically, we simply need to set $\mu_P = 0$ in Equation 3.31.

3.2.9 Duration and Expected Shortfall

Some of the problems with VaR can be solved by using a different measure of risk, called the expected shortfall. This measure of risk answers the following question: How large can we expect the loss of a portfolio to be when it is higher than VaR? As mentioned in point 3 in the above Subsection 3.2.8.1, the VaR measure does not say anything about the tails of the statistical distribution. This is an especially important problem when the underlying risk factor has a fat-tailed distribution, as shown in Figure 3.3, or when the portfolio contains highly nonlinear derivative securities, as we will see in later chapters.

Definition 3.7 *The* **expected shortfall** *is the expected loss on a portfolio P over the horizon T conditional on the loss being larger than the $(100-\alpha)$%, T VaR:*

$$\textit{Expected shortfall} = E\left[L_T | L_T > VaR\right] \tag{3.32}$$

For instance, a $100 million portfolio may have a 95%, 1-month expected shortfall of $4.28 million, meaning that when a bad event hits (losses higher than VaR), the portfolio stands to lose $4.28 million in average.

The calculation of expected shortfall is only slightly more involved than the one of VaR. For instance, for normally distributed variables, we have the following result:

Fact 3.7 *Consider Fact 3.6. Under these assumptions:*

$$\textit{95\% Expected shortfall} = -\left(\mu_P - \sigma_P \times \frac{f(-1.645)}{\mathcal{N}(-1.645)}\right) \tag{3.33}$$

$$= -\left(\mu_P - \sigma_P \times 2.0628\right) \tag{3.34}$$

where $f(x)$ denotes the standard normal density and $\mathcal{N}(x)$ is the standard normal cumulative density,[5] *The 99% expected shortfall is obtained as in Equation 3.34 except with the number "2.6649" in place of "2.0628."*

A quick comparison of Equations 3.34 and 3.31 shows that for the normal distribution case, the expected shortfall contains the same information as the Value-at-Risk, as the only difference is the coefficient that multiplies σ_P. But this is in fact exactly the reason for a new measure of risk: The expected shortfall is very useful precisely for those situations in which the portfolio losses are not expected to be normally distributed.

EXAMPLE 3.8

Consider again Example 3.7. The 95%, 1-month expected shortfall is easily computed in the case of a normal distribution, as we must simply change the coefficient "1.645" that multiplies σ_P with the coefficient "2.0628" (and similarly for the 99% expected shortfall). Given $\mu_P = -.0033$ and $\sigma_P = 2.0767$, we obtain

$$\text{(Normal disttribution approach):}\quad 95\%\ \text{ES} = \$4.2871\ \text{mil};\quad 99\%\ \text{ES} = \$5.5374\ \text{mil} \tag{3.35}$$

The numbers are quite different for the case in which the historical distribution approach is used. How do we compute the expected shortfall in this case? The methodology is just a slight modification of the VaR computation. In the VaR case, we first rank all of the portfolio P&L realizations under the various interest rate scenarios from the worst to the best, and then pick the 5% worst case. For the expected shortfall, we take the average of all of the realizations below the 5% worst case. A similar precedure is used for the 1% expected shortfall calculation. In this case, we obtain:

$$\text{(Historical distribution approach):}\quad 95\%\ \text{ES} = \$5.0709\ \text{mil};\quad 99\%\ \text{ES} = \$9.3344\ \text{mil} \tag{3.36}$$

We note in particular that the 99% expected shortfall is substantially larger under the historical distribution approach than under the normal distribution approach. This finding is a reflection of the fat-tailed distribution that characterizes the interest rate changes, and thus of the P&L dP, as shown in the bottom panel in Figure 3.3. In particular, extreme portfolio realizations occur more frequently than according to the normal distribution. It is worth pointing out that in contrast the VaR measure does not capture well the risk embedded in the tails of the distribution. For instance, in Example 3.7 the 99% VaR is $5.52 million, which is higher than the figure obtained under the normal distribution approach ($4.83 million), but not much higher. The expected shortfall is much better able to capture the risk from tail events.

3.3 INTEREST RATE RISK MANAGEMENT

Interest rate risk management is a key activity for banks, bond portfolio managers, corporations, governments, and, in fact, households. To understand the risks in interest rate fluctuations, consider the following example.

[5] That is, $f(x) = 1/\sqrt{2\pi} \times e^{-x^2/2}$ and $\mathcal{N}(x) = \int_{-\infty}^{x} f(y)dy$.

EXAMPLE 3.9

Ms. Caselli retired at the age of 60, with $1,000,000 in her retirement account. She now has to decide where to invest this amount of money for the next, say, 30 years. Treasury bonds are the only type of security she would consider, given her age. Should she invest in long-term bonds or short-term bonds? Consider the two extremes:

1. Invest all of $1,000,000 in 6-months T-bills.
2. Invest all of $1,000,000 in 30-year T-bonds.

What is the difference between these two strategies? If Ms. Caselli is going to consume only the interest on her investment, strategy 1 is more risky than strategy 2. Indeed, under strategy 1, fluctuations in interest rates imply fluctuations in the amount of money available for consumption. For instance, an interest rate change from 4% to 1% decreases Ms. Caselli's annual interest income from $40,000 to $10,000, a rather dramatic change. Instead, assuming that the 30-year bond sells at par and that the coupon rate is 4%, strategy 2 provides a certain $40,000 per year for all 30 years.

Most likely, Ms. Caselli is interested in using up some of her capital for consumption purposes. Indeed, cashing nothing but interest income may not produce enough funds on which to survive. If cash flow comes from the amount of capital available, the sensitivity of capital itself to interest rates becomes a big issue. For instance, look again at strategy 2. Assume that the zero coupon yield curve is flat at 4% (semi-annually compounded), so that a 30-year T-bond with 4% coupon trades at par. Such coupon bond has a duration of 17.72. Consider now an interest rate increase of 3% from 4% to 7% (as happened, for instance, in 1994). The capital losses on the investment would be approximately

$$\text{Capital losses} \approx 17.72 \times \$1 \text{ million } \times .03 = \$531,000$$

That is, a 3 percent increase in the interest rate may more than halve the savings of Ms. Caselli. If Ms. Caselli is not planning to consume out of her capital, this capital loss is of no consequence: She still possesses the same bond as before, which will keep paying the same $40,000 per year. But if she is planning to use up some of the capital for consumption, this strategy is clearly risky.

This example illustrates how the type of interest rate risk management that an institution or a person may want to engage in depends on the goals of the institution or individual.

3.3.1 Cash Flow Matching and Immunization

Ms. Caselli, in Example 3.9, can purchase an annuity from a financial institution. For instance, the financial institution may agree to pay $28,767 every six months for 30 years in exchange for the $1,000,000 deposit. Where is this number coming from? Assuming a flat term structure at a semi-annually compounded interest rate of 4%, the present value of this stream of cash flows is about $1,000,000:

$$\$1,000,000 = \$28,767 \times \sum_{i=1}^{60} \frac{1}{\left(1+\frac{4\%}{2}\right)^i} \tag{3.37}$$

where 60 is the number of payments.

How can the financial institution now hedge this commitment to pay exactly $28,767 twice a year for 30 years? What risks does it take?

1. **Cash Flow Matching.** The financial institution can purchase a set of securities that pays exactly $28,767 every six months. For instance, it can purchase 60 zero coupon bonds, each with a $28,767 face value, and with maturities of 6 months, 1 year, 1.5 years, and so on up to 30 years. The value of these securities is, by construction, equal to $1,000,000, as the present value in Equation 3.37 applies to this case. One drawback of this strategy, though, is that the financial institution should find exactly the type of securities that are required for the cash flow matching, such as the sequence of zero coupon bonds with $28,500 face value at 6-month intervals. Such an endevour may be problematic, and costly, as many securities are nonliquid.

2. **Immunization.** The financial institution can engage in a dynamic immunization strategy. Such a strategy involves the choice of a portfolio of securities with the *same present value and duration of the cash flow commitments to pay*. Immunization is preferred over cash flow matching as it allows the institution to choose bonds that have favorable properties in terms of liquidity and transaction costs. If executed properly, this method generates the desired stream of cash flows.

While the cash flow matching is relatively straightforward, it is instructive to work through an example illustrating the immunization strategy. We continue with Example 3.9.

■ **EXAMPLE 3.10**

The financial institution that took up the commitment to pay $28,767 every six months can ensure the ability to pay by engaging in the following dynamic strategy. Let $x_t\%$ denote the fraction of the total capital – $1,000,000 at initiation – invested in the 4%, 30-year bond, as described in Example 3.9. Assume that the remaining $(1 - x_t)\%$ is kept as cash in a deposit account, thereby yielding the overnight deposit rate. The duration of the annuity promised to Ms. Caselli is about 12.35. The 30-year coupon bond has a duration of 17.72, while the overnight deposit has zero duration, as the deposit rate resets daily. Because the immunization strategy calls for equating the duration of the portfolio with the one of the annuity, it then requires that at time 0:

$$x_0\% \times 17.72 + (1 - x_0\%) \times 0 = 12.34 \Longrightarrow x_0 = 71\%$$

Assume that the financial institution rebalances every six months. Then, at any time $t = .5, 1, 1.5, ..., 30$ the financial institution:

- collects the $\frac{4\%}{2}$ coupon from the 30-year bond;
- collects the interest cumulated over the six months on the cash deposit;
- pays the annuity cash flow of $28,767 to Ms. Caselli; and
- reinvests the remaining balance in long-term bonds and overnight deposit according to the rule:

$$\text{Percentage investment in long-term bond} = x_t = \frac{\text{Duration of annuity}}{\text{Duration of long-term bond.}} \tag{3.38}$$

Table 3.3 illustrates the strategy. Column (1) reports the time at which coupon payments are made, and rebalancing takes place. For convenience, assume that annuity payments and the long-term bond coupon payments occur on the same date. Column (2) reports a possible path of interest rates, from 4% to 11% and down again to 8% in the course of 30 years. These interest rates have been simulated. Column (3) computes the balance of the financial institution. It starts out with $1 million, and then the balance declines as the financial institution makes coupon payments to Ms. Caselli. We are more explicit about the information in this column below. Column (4) reports the present value of the annuity, assuming that the term structure is flat and equal to the interest rate in Column (2). Column (5) indicates the duration of the annuity. Note that both the present value and the duration of the annuity tend to decline over time. Columns (6) and (7) report the present value and duration, respectively, of the 4%, 30-year T-bond that is used in the immunization strategy. Column (8) reports the fraction of capital x_t invested in the 30-year bond, obtained from using Equation 3.38. Column (9) shows the total cash obtained at the end of each six-month period t, from the investment in overnight deposits at the beginning of the period. That is,

$$\text{Interest payment [column (9)]} = W_t \times (1 - x_t) \times r_t/2 \tag{3.39}$$

Similarly, Column (10) represents the total coupon received from the 30-year bond investment

$$\text{Coupon payment [column (10)]} = \frac{W_t \times x_t}{\text{Price T-bond in Column (6)}} \times 4\%/2 \tag{3.40}$$

Finally, returning to Column (3), the total amount of capital at the institution is updated by taking into account inflows and outflows. That is

$$\begin{aligned} W_{t+1} &= W_t \times (1 - x_t) + W_t \times x_t \times \text{Capital gain on T-bond} && (3.41)\\ &\quad + \text{Interest in (9) at } t + \text{coupon in (10)} && (3.42)\\ &\quad - \text{Annuity coupon (\$28,767)} && (3.43) \end{aligned}$$

Notice from the last row in Table 3.3 that the strategy still leaves $69,375 at maturity. If the interest rate was constant for the overall period and equal to 4%, then the final amount of wealth W_T would be exactly zero. There is a reason why the final wealth came up positive – due to the convexity of bond prices with respect to interest rates – that we discuss in Chapter 4.

Was this luck? That is, if we consider a different path of interest rates, would we still get a positive number? Or are there instances in which the final wealth was negative? To convince ourselves that this dynamic immunization strategy works, we can repeat the above exercise many times, for many interest rate scenarios, and plot the histogram of the final value W_T. This is done in Figure 3.4. As it can be seen, the strategy works well, as the final wealth is always positive.

3.3.2 Immunization versus Simpler Investment Strategies

How does the immunization strategy compare to other simpler strategies, such as investing fixed proportions in the long-term T-bond and cash? The panels in Figure

Table 3.3 Example of Immunization Strategy

(1) t	(2) r_2	(3) W_t	(4) PV Annuity	(5) D Annuity	(6) PV T-bond	(7) D T-bond	(8) x_t	(9) Interest Payment	(10) Coupon T-bond
0.5	4.00%	1,000,000.00	1,000,000.00	12.35	1.00	17.73	0.70	6,069.48	13,930.52
1.0	4.53%	931,694.92	931,527.35	11.84	0.91	17.00	0.70	6,394.58	14,198.57
1.5	5.46%	834,369.76	833,021.75	11.12	0.79	15.88	0.70	6,818.28	14,817.53
2.0	5.80%	798,630.50	797,418.94	10.79	0.75	15.42	0.70	6,949.54	14,901.35
2.5	5.07%	858,995.30	855,333.53	11.08	0.84	16.03	0.69	6,727.23	14,125.61
3.0	5.70%	798,169.63	794,486.32	10.59	0.77	15.31	0.69	6,997.57	14,430.69
3.5	5.97%	770,555.63	767,029.58	10.32	0.74	14.94	0.69	7,111.90	14,430.83
4.0	5.51%	802,061.06	797,212.37	10.43	0.79	15.24	0.68	6,975.91	13,867.73
4.5	5.75%	775,789.15	771,099.47	10.17	0.76	14.90	0.68	7,094.48	13,838.82
5.0	5.62%	780,072.25	774,970.46	10.10	0.78	14.89	0.68	7,054.71	13,518.96
5.5	5.41%	789,534.15	783,823.66	10.05	0.81	14.92	0.67	6,967.50	13,154.64
6.0	4.44%	863,655.83	853,748.21	10.34	0.93	15.51	0.67	6,387.38	12,332.29
6.5	3.84%	909,537.36	896,984.46	10.45	1.03	15.76	0.66	5,880.42	11,770.13
7.0	4.37%	851,902.96	839,884.97	10.04	0.95	15.19	0.66	6,309.03	11,906.61
7.5	4.85%	803,888.62	792,339.74	9.68	0.88	14.69	0.66	6,648.02	11,997.12
8.0	5.22%	767,558.29	756,355.88	9.37	0.84	14.28	0.66	6,882.78	12,008.02
8.5	5.80%	720,606.80	709,848.70	9.00	0.78	13.75	0.65	7,219.58	12,133.76
9.0	6.21%	688,142.41	677,682.96	8.71	0.74	13.35	0.65	7,429.60	12,138.47
9.5	7.10%	633,131.27	623,058.85	8.27	0.66	12.70	0.65	7,834.29	12,422.01
10.0	7.90%	589,160.97	579,458.13	7.90	0.61	12.14	0.65	8,132.71	12,628.90
10.5	8.77%	547,447.66	538,080.12	7.52	0.55	11.57	0.65	8,401.06	12,856.82
11.0	8.00%	574,852.24	563,207.45	7.61	0.61	11.84	0.64	8,227.02	12,148.50
11.5	8.34%	555,509.40	543,902.64	7.39	0.59	11.56	0.64	8,354.68	12,038.36
12.0	7.91%	567,397.43	554,342.75	7.36	0.62	11.62	0.63	8,215.15	11,532.52
12.5	7.96%	559,108.10	545,629.81	7.21	0.62	11.45	0.63	8,233.98	11,269.64
13.0	8.59%	529,197.45	516,057.51	6.93	0.59	11.05	0.63	8,481.00	11,279.49
13.5	9.55%	491,704.85	479,048.80	6.59	0.54	10.55	0.62	8,813.42	11,424.64
14.0	9.27%	495,447.31	481,641.25	6.52	0.56	10.52	0.62	8,733.50	10,977.17
14.5	10.09%	465,514.05	452,081.69	6.23	0.52	10.11	0.62	9,009.18	11,013.67
15.0	10.49%	449,693.51	436,142.65	6.04	0.51	9.86	0.61	9,135.49	10,837.50
15.5	10.19%	452,268.71	437,477.20	5.95	0.53	9.80	0.61	9,046.62	10,380.01
16.0	10.10%	448,755.10	433,060.02	5.83	0.54	9.67	0.60	9,002.83	10,010.52
16.5	10.84%	425,079.56	409,594.85	5.59	0.51	9.33	0.60	9,243.38	9,931.53
17.0	11.34%	408,561.98	392,937.71	5.39	0.50	9.06	0.59	9,387.23	9,743.14
17.5	12.08%	388,211.13	372,656.40	5.16	0.48	8.74	0.59	9,600.17	9,615.89
18.0	11.95%	385,320.32	368,625.65	5.04	0.49	8.60	0.59	9,532.93	9,203.05
18.5	11.58%	386,506.22	368,186.27	4.94	0.52	8.49	0.58	9,356.17	8,733.27
19.0	11.45%	382,431.96	362,838.04	4.81	0.53	8.32	0.58	9,237.02	8,340.71
19.5	11.63%	372,563.61	352,126.89	4.64	0.53	8.08	0.57	9,218.34	8,024.12
20.0	11.94%	360,381.56	339,248.28	4.46	0.53	7.82	0.57	9,242.82	7,733.61
20.5	11.82%	354,983.36	332,387.26	4.32	0.55	7.60	0.57	9,072.49	7,350.17
21.0	11.69%	349,068.99	324,921.21	4.16	0.57	7.37	0.57	8,874.41	6,973.10
21.5	11.35%	345,393.43	319,247.45	4.02	0.59	7.13	0.56	8,556.37	6,571.90
22.0	11.48%	334,579.86	307,161.34	3.84	0.60	6.84	0.56	8,426.27	6,249.34
22.5	11.35%	326,472.31	297,276.29	3.67	0.62	6.56	0.56	8,158.47	5,895.98
23.0	10.98%	320,443.04	288,950.85	3.51	0.65	6.26	0.56	7,745.83	5,523.33
23.5	10.15%	318,131.85	283,432.61	3.35	0.70	5.97	0.56	7,078.31	5,123.72
24.0	10.30%	303,990.10	267,790.08	3.15	0.71	5.62	0.56	6,876.17	4,825.62
24.5	10.72%	287,314.17	249,904.43	2.94	0.71	5.25	0.56	6,770.09	4,545.18
25.0	10.32%	276,993.67	236,955.12	2.75	0.74	4.89	0.56	6,252.49	4,213.72
25.5	9.51%	268,191.16	224,811.75	2.56	0.78	4.51	0.57	5,521.92	3,875.08
26.0	9.97%	249,375.96	204,603.79	2.34	0.79	4.11	0.57	5,359.27	3,603.08
26.5	10.15%	232,116.30	185,357.01	2.12	0.80	3.70	0.57	5,027.23	3,320.09
27.0	9.54%	217,795.73	167,873.19	1.91	0.84	3.28	0.58	4,343.35	3,021.95
27.5	10.38%	196,476.08	145,132.50	1.68	0.84	2.84	0.59	4,180.27	2,763.55
28.0	10.79%	176,812.76	123,208.88	1.45	0.85	2.39	0.60	3,771.06	2,501.20
28.5	10.12%	158,948.97	101,873.37	1.22	0.89	1.94	0.63	2,981.69	2,242.73
29.0	9.64%	138,908.31	78,608.20	0.98	0.92	1.47	0.67	2,210.22	2,016.28
29.5	8.99%	117,422.88	53,876.71	0.74	0.95	0.99	0.75	1,308.69	1,852.74
30.0	8.30%	94,234.40	27,622.23	0.50	0.98	0.50	1.00	0.00	1,924.37
		69,375.15			1.00				

Figure 3.4 Performance Immunization Strategy in Simulations

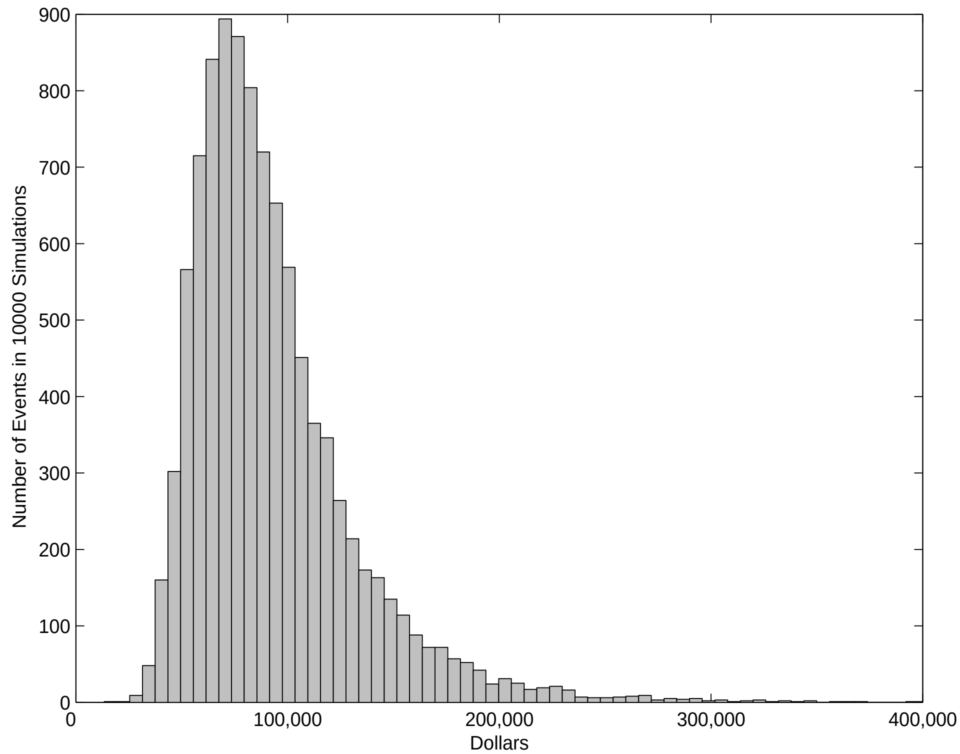
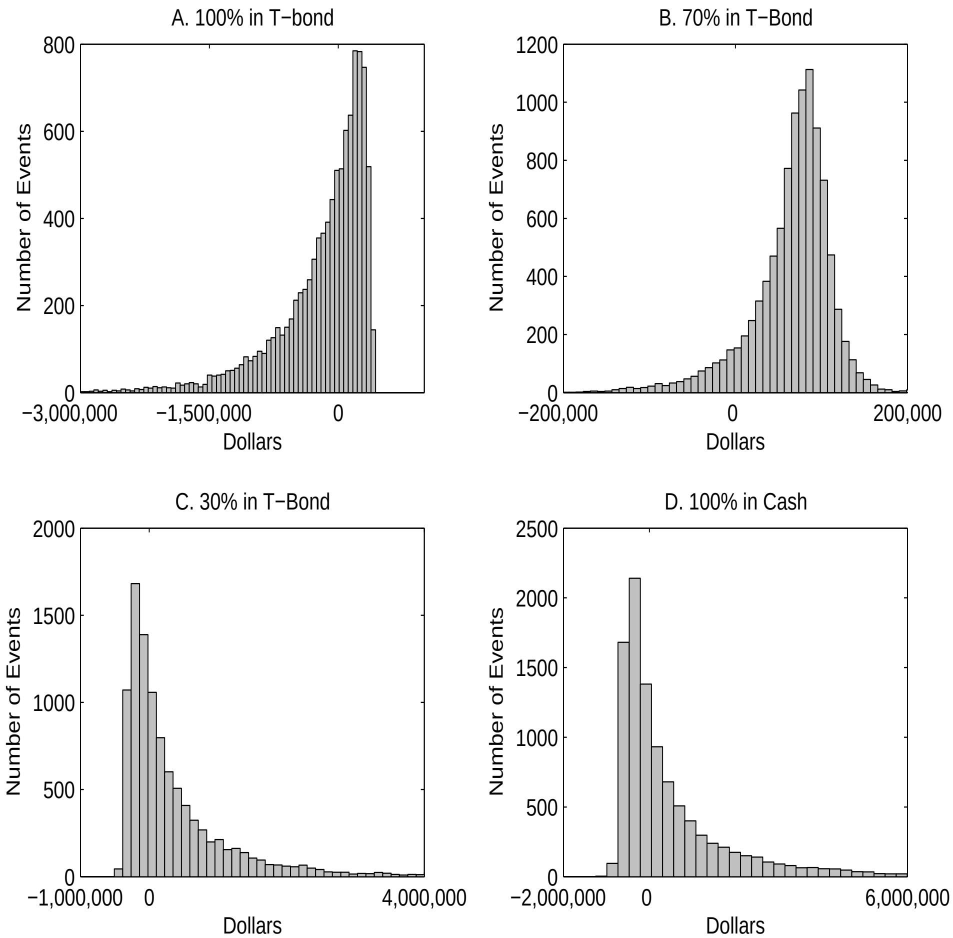


Figure 3.5 Performance Fixed Investment Strategy in Simulations



3.5 plot the results of an analysis similar to the immunization strategy in Figure 3.4, but in which the investment in long-term T-bond x_t is kept fixed at 100% (Panel A), 70% (Panel B), 30% (Panel C), 0% (Panel D). The figure clearly shows that in these cases, the bank stands to lose money with some probability. For instance, if the financial institution was to keep 100% invested in the long term T-bond, it would stand to lose money about 50% of the time. Instead, the constant strategy of 70% and 30% in long term bond stand to lose money about 10% and 40% of the time, respectively. On the other hand, a 100% investment in cash only (Panel D) also is not appropriate, as the strategy loses money again about 50% of the time.

3.3.3 Why Does the Immunization Strategy Work?

What is the intuition behind immunization strategies? Why do they work? Think again about the two extremes: 100% investment in long-term bonds loses money when interest rates go up, because bond prices decline when interest rates increase. Similarly, a 100%

Table 3.4 The Duration Mismatch

	Assets	Liabilities
Commercial Banks	Long-term loans (High D)	Deposits (Low D)
Insurance Companies	Short-Term T-Bonds (Low D)	Long-Term Commitments (High D)
Pension Plans	Medium-Term T-Bonds (Low D)	Long-Term Commitments (High D)
Corporations	Long-Term Receivables (High D)	Floating Rate Bonds (Low D)

investment in cash loses money when interest rates go down. If the interest rate goes to zero, for instance, then there is not enough of capital to make up the annuity coupon. Clearly, the safer strategy is in the middle. Indeed, the immunization strategy effectively ensures that the losses on the cash investment due to a declining interest rates are compensated by the capital gains on the long-term bond. The duration enters the picture here, as it measures the sensitivity of bond prices to interest rate changes.

3.4 ASSET-LIABILITY MANAGEMENT

Asset liability management is the most classic example of interest rate risk management. Many financial institutions have a duration mismatch between their assets and their liabilities. For instance, a commercial bank collects deposits – a short-term liability whose interest rate changes daily – to make medium- and long-term loans to other business or households. If the medium- and long-term loans have fixed coupons, as in fixed rate mortgages for instance, then the duration of the assets is relatively long, for instance 5 years or more. On the other hand, deposits have a duration close to zero, as the short term interest rate needs to be adjusted frequently as market conditions change.

What happens if there is a hike in interest rates?

The analysis in previous sections shows that the value of the assets drop, while the value of the liabilities does not change. In flow terms, the bank now has to pay a high rate on the deposits, but still receives a low coupon from its assets. In essence, the bank is in trouble.

The duration analysis in the previous sections can be applied more generally to analyze the relative potential duration mismatch between assets and liabilities. One important problem is that financial institutions have very complex asset composition. However, quite independent of the types of assets, it is possible to compute the duration of the overall portfolio of assets. Indeed, we can consider the total assets of the firm as a portfolio of securities (e.g. individual loans, receivables, and so on) and thus use the earlier formula in Equation 3.8 to compute the duration of assets as a weighted average of the durations of its components. For instance, if a firm has n individual loans, whose values are A_1, A_2,...,A_n and their durations are $D_{A,1}$, $D_{A,2}$,..., $D_{A,n}$, then we can compute the duration of assets as

$$\text{Duration of assets } D_A = \sum_{i=1}^{n} w_{A,i} D_{A,i}$$

where

$$w_{A,i} = \frac{A_i}{\sum_{i=1}^{n} A_i}$$

Similarly, financial institutions also have very complex liabilities, as they do not finance their loans only with deposits, but also with longer-term vehicles (e.g., certificates of deposit), long-term bonds, and, of course, equity. In the same fashion as with assets, the financial institution can consider its liabilities as a portfolio and compute the duration of liabilities. Denoting L_1, L_2,...,L_m the current value of each of its m liabilities (excluding equity), and $D_{L,1}$, $D_{L,2}$,...,$D_{L,m}$ their durations, we obtain

$$\text{Duration of liabilities } D_L = \sum_{i=1}^{n} w_{L,i} D_{L,i}$$

where

$$w_{L,i} = \frac{L_i}{\sum_{i=1}^{n} L_i}$$

The aim of asset - liability management is often taken to minimize the impact that the variation in the level of interest rates has on the value of equity. Since equity E is given by total assets (A) minus total liability (L),

$$E = A - L$$

we have that duration mismatch occurs whenever $D_E \neq 0$. Treating equity as a portfolio, we obtain

$$D_E = \frac{A}{A-L} \times D_A - \frac{L}{A-L} \times D_L \tag{3.44}$$

Therefore, $D_A^{\$} = AD_A \neq LD_L = D_L^{\$}$ results in a duration mismatch problem, and variation of interest rates affect the value of equity.

EXAMPLE 3.11

Consider a hypothetical financial institution mainly engaged in making long-term loans. The balance sheet of such financial institution may look like the one in Table 3.5. Total assets are around \$2.4 billion, with a dollar duration of \$19.74 billion. Total liabilities are \$1.8 billion with a dollar duration of only \$5 billion. As a consequence, the market value of equity is \$600 million, but with a dollar duration of \$14.740 billion. The implication of this mismatch is that a parallel upward shift in interest rates of 1% generates a decline in assets far greater than in liabilities, implying an equity decline of $\$147.4$ million. In percentage, this corresponds to a 24%decline in market value of equity.

To reduce or eliminate this maturity mismatch, the financial firm may alter the composition of its portfolio. One possibility is to issue long-term debt to increase the duration of liabilities. Intuitively, if interest rates increase the financial institution gains from making coupon payments on its long term debt that are below the current rate. Another far more common possibility is to use derivative securities, such as swaps, to alter the duration of assets. We explore further this methodology in Chapter 5, after we cover the properties of swaps and other derivative securities.

3.5 SUMMARY

In this chapter we covered the following topics:

Table 3.5 Asset and Liabilities of a Financial Institution

Assets				Liabilities			
Item	Amount	Duration	Dollar Duration	Item	Amount	Duration	Dollar Duration
Cash	100	0	0	Deposits	600	0	0
S.T. Loans	300	0.8	240	S.T. Debt	400	0.5	200
M.T. Loans	500	3	1500	M.T. Debt	400	4	1600
L.T. Loans	1500	12	18000	L.T. Debt	400	8	3200
Total	2400		19740	Total	1800		5000
				Equity	600		14740

1. Duration: The (negative of the) percentage sensitivity of a security to parallel shift in the term structure of interest rates is known as duration. As an example, the duration of zero coupon bonds is just their time to maturity.

2. Duration of a portfolio of securities: This can be computed as the weighted average of durations of the individual securities in the portfolio, where the weights equal the percentage holdings of the securities.

3. Dollar duration: Unlike duration, the dollar duration measures the (negative of the) dollar changes in prices due to a parallel shift in the term structure of interest rates. This can be used for securities or strategies that require a zero investment.

4. Value-at-Risk: VaR is a risk measure that computes the maximum losses a portfolio can sustain, within a given horizon, with a given probability. For instance, a 95%, one month VaR provides the maximum loss a portfolio sustains with 95% probability.

5. Expected Shortfall: A risk measure that computes the expected losses on a portfolio, conditional on these losses being larger than VaR, expected shortfall is a measure that is better able to deal with tail events than VaR.

6. Immunization: Immunization is a strategy to make a portfolio insensitive to changes in interest rate.

7. Asset-Liability Management: This is a strategy of choosing the (dollar) duration of liabilities to match the (dollar) duration of assets. It helps reduce the sensisitivy of equity to changes in interest rates, and ensures that cash flows received from assets are sufficient to pay the cash flows from liabilities.

3.6 EXERCISES

1. Today is May 15, 2000, and the current, semi-annually compounded yield curve is in Table 3.6. Compute the duration for the following securities:

   (a) 3-year zero coupon bond

   (b) 3 1/4-year coupon bond paying 6% semiannually

   (c) 1-year coupon bond paying 4% quarterly

Table 3.6 Yield Curve on May 15, 2000

Maturity	Yield	Maturity	Yield	Maturity	Yield
0.25	6.33%	2.75	6.86%	5.25	6.39%
0.50	6.49%	3.00	6.83%	5.50	6.31%
0.75	6.62%	3.25	6.80%	5.75	6.24%
1.00	6.71%	3.50	6.76%	6.00	6.15%
1.25	6.79%	3.75	6.72%	6.25	6.05%
1.50	6.84%	4.00	6.67%	6.50	5.94%
1.75	6.87%	4.25	6.62%	6.75	5.81%
2.00	6.88%	4.50	6.57%	7.00	5.67%
2.25	6.89%	4.75	6.51%	7.25	5.50%
2.50	6.88%	5.00	6.45%	7.50	5.31%

Notes: Yields are calculated based on data from CRSP (Daily Treasuries).

(d) 6-year floating rate bond with a zero spread, paying semiannually

(e) 3-year floating rate bond with a 35 basis point spread, paid semiannually

(f) 4 1/4 year floating rate bond with 50 basis point spread, paid semiannually

2. An investor is planning a $100 million short-term investment and is going to choose among two different portfolios. This investor is seriously worried about interest rate volatility in the market. Compute the duration of the portfolios. Which one is more adequate for the investor's objective? Assume today is May 15, 2000, which means you may use the yield curve presented in Table 3.6

   Portfolio A

   - 40% invested in 4 1/4-year bonds paying 5% semiannually
   - 25% invested in 7-year bonds paying 2.5% semiannually
   - 20% invested in 1 3/4-year floating rate bonds with a 30 basis point spread, paying semiannually
   - 10% invested in 1-year zero coupon bonds
   - 5% invested in 2-year bonds paying 3% quarterly

   Portfolio B

   - 40% invested in 7-year bonds paying 10% semiannually
   - 25% invested in 4 1/4-year bonds paying 3% quarterly
   - 20% invested in 90-day zero coupon bonds
   - 10% invested in 2-year floating rate bonds with zero spread, paying semiannually
   - 5% invested in 1 1/2 -year bonds paying 6% semiannually

3. Compute the Macaulay and modified duration for the same securities as in Exercise 1.

4. Using the yield curve in Table 3.6, compute the dollar duration for the following securities:

(a) Long a 5-year coupon bond paying 4% semiannually
(b) Short a 7-year zero coupon bond
(c) Long a 3 1/2-year coupon bond paying 7% quarterly
(d) Long a 2-year zero spread floating rate bond paid semiannually
(e) Short a 2 1/4-year zero spread floating rate bond paid semiannually
(f) Short a 5 1/4-year floating rate bond with a 25 basis point spread paid semiannually

5. The investor in Exercise 2 is still worried about interest rate volatility. Instead of a duration measure, the investor wants now to know the following:

   (a) What is the dollar duration of each portfolio?
   (b) What is PV01 for each portfolio?
   (c) Does the conclusion arrived at in Exercise 2 stand?

6. Due to a series of unfortunate events, the investor in Exercise 2 just found out that he must raise $50 million. The investor decides to short the long-term bonds in each portfolio to raise the $50 million. In other words, for portfolio A the investor would spend the same on all securities except for the 7-year coupon bonds (paying 2.5% semiannually) from which the investor will short enough to get to $50 million. For portfolio B the investor would spend the same on all other securities except for the 7-year coupon bonds (paying 10% semiannually) from which the investor will short enough to get to $50 million.

   (a) How many bonds of each kind does the investor have to short?
   (b) What is the new dollar duration of each portfolio?
   (c) Does the conclusion arrived at in Exercise 2 stand?

Exercises 7 to 12 use the two yield curves at two moments in time in Table 3.7, and the following portfolio:

- Long $20 million of a 6-year inverse floaters with the following quarterly coupon:

$$\text{Coupon at } t = 20\% - r_4(t - 0.25)$$

  where $r_4(t)$ denotes the quarterly compounded, 3-month rate.
- Short $20 million of 4-year floating rate bonds with a 45 basis point spread paying semiannually
- Short $30 million of a 5-year zero coupon bond

7. You are standing on February 15, 1994 and you hold the following portfolio:

   (a) What is the total value of the portfolio?
   (b) Compute the dollar duration of the portfolio.

8. You are worried about interest rate volatility. You decided to hedge your portfolio a 3-year coupon bond paying 4% on a semiannual basis.

Table 3.7 Two Term Structures of Interest Rates

Maturity	02/15/94 Yield (c.c.)	02/15/94 $Z(t,T)$	05/13/94 Yield (c.c.)	05/13/94 $Z(t,T)$
0.25	3.53%	0.9912	4.13%	0.9897
0.50	3.56%	0.9824	4.74%	0.9766
0.75	3.77%	0.9721	5.07%	0.9627
1.00	3.82%	0.9625	5.19%	0.9495
1.25	3.97%	0.9516	5.49%	0.9337
1.50	4.14%	0.9398	5.64%	0.9189
1.75	4.23%	0.9287	5.89%	0.9020
2.00	4.43%	0.9151	6.04%	0.8862
2.25	4.53%	0.9031	6.13%	0.8712
2.50	4.57%	0.8921	6.23%	0.8558
2.75	4.71%	0.8786	6.31%	0.8406
3.00	4.76%	0.8670	6.39%	0.8255
3.25	4.89%	0.8531	6.42%	0.8117
3.50	4.98%	0.8400	6.52%	0.7959
3.75	5.07%	0.8268	6.61%	0.7805
4.00	5.13%	0.8145	6.66%	0.7663
4.25	5.18%	0.8023	6.71%	0.7519
4.50	5.26%	0.7893	6.73%	0.7387
4.75	5.31%	0.7770	6.77%	0.7251
5.00	5.38%	0.7641	6.83%	0.7106
5.25	5.42%	0.7525	6.86%	0.6977
5.50	5.43%	0.7418	6.89%	0.6846
5.75	5.49%	0.7293	6.93%	0.6713
6.00	5.53%	0.7176	6.88%	0.6619

Notes: Yields are calculated based on data from CRSP (Daily Treasuries).

(a) How much should you go short/long on this bond in order to make it immune to interest rate changes?

(b) What is the total value of the portfolio now?

9. Assume that it is now May 15, 1994 and that the yield curve has changed accordingly (see Table 3.7).

   (a) What is the value of the unhedged portfolio now?

   (b) What is the value of the hedged portfolio?

   (c) Is the value the same? Did the immunization strategy work? How do you know that changes in value are not a product of coupon payments made over the period?

10. Instead of assuming that the change took place 6 months later, assume that the change in the yield curve occurred an instant after February 15, 1994.

    (a) What is the value of the unhedged portfolio?

    (b) What is the value of the hedged portfolio?

11. Now use the February 15, 1994 yield curve to price the stream of cash flows on May 15, 1994.

    (a) What is the value of the unhedged portfolio?

    (b) What is the value of the hedged portfolio?

12. From the answers to the Exercises 7 - 11, answer the following:

    (a) What is the change in value in the portfolio due to the change in time only, without change in interest rates?

    (b) Is this difference a loss?

    (c) Once we have adjusted for paid coupons, what is the change in value of the portfolio due to interest rate movements?

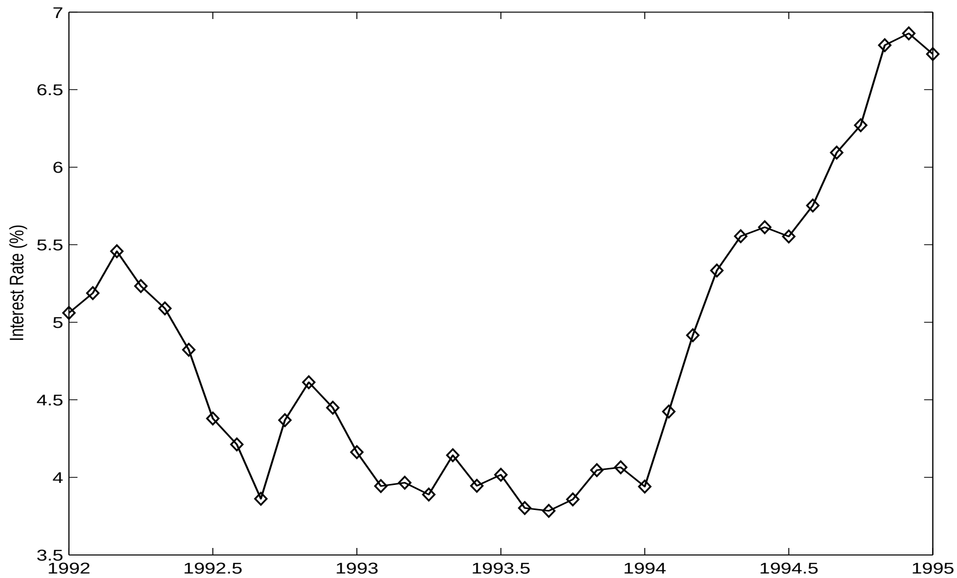
3.7 CASE STUDY: THE 1994 BANKRUPTCY OF ORANGE COUNTY

As discussed in Section 3.1.2, in 1994 Orange County lost $1.6 billion out of a portfolio of $7.5 billion in assets as a result of an unexpected increase in interest rates, from 3% to 5.7%.[6] Figure 3.6 shows the sudden steep increase in the level of interest rates in 1994.

The lessons learned in this chapter will help us understand what type of exposure Orange County had in its portfolio that could lead to a loss of this magnitude.

[6]This section and the next are based on publicly available information and they are only meant to illustrate the concepts introduced in this chapter. No claim of wrongdoing by any party is made here. Descriptive material is from the case study ERISK: Orange County, downloaded from http://www.erisk.com/Learning/CaseStudies/OrangeCounty.asp.

Figure 3.6 The Level of Interest Rates, 1992 - 1994



Data Source: CRSP.

3.7.1 Benchmark: What if Orange County was Invested in Zero Coupon Bonds Only?

A useful starting point is to suppose that Orange County portfolio was invested only in zero coupon bonds and then find the maturity of these bonds necessary to bring a loss of \$1.6 billion. We can do this by using the concept of duration introduced in Section 3.2. Recall that duration is defined as the (negative of the) sensitivity of a security, or a portfolio, to parallel shifts in the term structure of interest rates. More specifically, Equation 3.1 defines duration as

$$\text{Duration } = D_P = -\frac{1}{P}\frac{dP}{dr} \tag{3.45}$$

We know that the portfolio value before the hike in interest rates was approximately $\$7.5$ billion, and thus $P = 7.5$. In addition, the change in the level of interest rates was $dr = 6.7\% - 4\% = 0.027$. Finally, the loss was $dP = -1.6$. Substituting into Equation 3.45 we find

$$\text{Duration } = -\frac{1}{7.5}\frac{-1.6}{.027} = 7.90 \tag{3.46}$$

That is, if Orange County's portfolio was invested only in Treasury securities, given the ex-post losses, we would gather that the duration of his portfolio should have been around 7.90. From Equations 3.5 and 3.6, the duration of a zero coupon bond is equal to its time to maturity. Thus, Orange County would have been hit by the same type of losses as if all of its portfolio was invested in zero coupon bonds with 7.9 years to maturity.

3.7.2 The Risk in Leverage

Orange County's portfolio, however, was not only invested in Treasury securities. By using the repo market, Orange County effectively levered up the portfolio position to $20.5 billion (see Chapter 1 to review the repurchase agreements contract.) Essentially, the Treasurer of Orange County could pledge the portfolio's highly liquid Treasury securities as collateral in a repo transaction so as to obtain other funds to invest further in Treasuries or other securities.

For now, we only consider the effect of leverage. First, the duration of the levered portfolio now has to be computed using the assets and liability formula in Equation 3.44, that is

$$\text{Duration leveraged portfolio} = w_A \times D_A + w_L \times D_L \qquad (3.47)$$

where $w_A = A/(A-L) = 20.5/7.5$ and $w_L = -L/(A-L)$. The liability is given by the repo transactions, which are financed at the overnight rate. The duration of liabilities, thus, is approximately zero. That is, $D_L \approx 0$. The duration of assets, in contrast, is given by

$$D_A = -\frac{1}{20.5}\frac{-1.6}{.027} = 2.89 \qquad (3.48)$$

Of course, $w_A \times D_A = 7.90$ as before. The point of this calculation, however, is that the portfolio of Orange County may well have been invested only in short-term zero coupon bonds with maturity 2.89. Although such an investment per se appears very safe, the presence of the large leveraged position generates a much higher duration of the leveraged portfolio itself, which could lead to the $1.6 billion losses when the interest rates moved by 2.7%, as they did in 1994.

3.7.3 The Risk in Inverse Floaters

Although the main reason for having a record-breaking loss in the Orange County portfolio was leverage, the trigger was the change in interest rates and its effect on inverse floaters. Recall from Chapter 2 that inverse floaters have a coupon that moves inversely to short-term floating rates. This implies that when interest rates go up, the price of inverse floaters receive a negative shock from two channels:

1. The discount channel: If interest rates go up, prices of zero coupon bonds fall as future cash flows are worth less in today's money.

2. The cash flow channel: If interest rates go up, the actual cash flow is reduced because coupon payments move inversely to interest rates.

The sensitivity of inverse floaters to interest rates can be calculated through the concept of duration. How can we compute the duration of an inverse floater? Here, we must remember that an inverse floater is given by a portfolio of more basic securities, of which we can compute the duration easily.

In particular, recall from Chapter 2, Equation 2.43, that we can write the price of a (plain vanilla) inverse floater with maturity T with annual payments and coupon $c(t) = \overline{c} - r_1(t-1)$ as

$$\text{Price inverse floater } P_{IF}(0,T) = P_z(0,T) + P_c(0,T) - P_{FR}(0,T) \qquad (3.49)$$

Table 3.8 The Duration of the 15% Fixed Rate Bond

Date	Cash Flow	Discounted Cash Flow	Weight w	T	$w * T$
12/31/1994	0.15	0.1446	0.1123	1	0.1123
12/31/1995	0.15	0.1379	0.1070	2	0.2141
12/31/1996	1.15	1.0057	0.7807	3	2.3421
	Total Value	1.2884		Duration:	2.6685

where $P_z(0,T)$ is the price of a zero coupon bond, $P_c(0,T)$ is the price of a $\overline{c}$ coupon bond, and $P_{FR}(0,T)$ is the price of a floating rate bond, all of them with maturity T. Thus, we can compute the duration of the inverse floater by applying the formula for the duration of a portfolio, namely, Equation 3.10.

We consider here the simple case discussed in Chapter 2, $\overline{c} = 15\%$, $T = 3$ and coupon payments are annual. In this case, we obtained $P_{IF}(0,3) = \$116.28$, $P_z(0,3) = 87.45$, $P_c(0,3) = 128.83$, and $P_{FR}(0,3) = 100$. The duration of the inverse floater can be computed then as

$$D_{Inverse} = w_{Zero} \times D_{zero} + w_{Fixed} \times D_{Fixed} + w_{Floating} \times D_{Floating} \tag{3.50}$$

where $w_{Zero} = P_z(0,3)/P_{IF}(0,3) = 0.7521$, $w_{Fixed} = P_c(0,3)/P_{IF}(0,3) = 1.1079$, and $w_{Floating} = -P_{FR}(0,3)/P_{IF}(0,3) - .8600$ are the weights. The duration of a zero coupon bond equals its time to maturity, thus $D_{zero} = 3$. The duration of a floating rate bond with annual coupons is equal to the time of the first coupon at reset dates. Thus, $D_{Floating} = 1$. The only term left to calculate is the duration of the fixed rate bond D_{Fixed}. Table 3.8 performs the computation, obtaining $D_{Fixed} = 2.6684$.

We can substitute everything into Equation 3.50, to find

$$\begin{aligned} D_{Inverse} &= w_{Zero} \times D_{zero} + w_{Fixed} \times D_{Fixed} + w_{Floating} \times D_{Floating} \\ &= 0.7521 \times 3 + 1.1079 \times 2.6685 - .8600 \times 1 \\ &= 4.35 \end{aligned}$$

The duration of the 3-year inverse floater is 4.35. It is important to note that the duration is higher than the inverse floater maturity (3 years). In this sense, the notion of "duration" as a temporal average of cash flows plays no role here, as we are interpreting the duration as the sensitivity of the security's price to changes in interest rates. Depending on how cash flows move with interest rates, this sensitivity can be larger or smaller than the maturity of the security itself.

3.7.4 The Risk in Leveraged Inverse Floaters

Recall that a leveraged inverse floater has a coupon that moves (inversely) to interest rates by more than one-to-one. For instance, the leveraged inverse floater discussed in Section 2.8.4 of Chapter 2 has a coupon

$$c(t) = 25\% - 2 \times r_1(t-1) \tag{3.51}$$

Table 3.9 The Duration of the Leverage Inverse Floater

Security	Value	Weight w	Duration D	$D * w$
$2 \times P_z(3)$	174.91	1.3320	3.00	3.9959
$P_c(3)$	156.41	1.1911	2.5448	3.0311
$-2 \times P_{FR}(3)$	-200.00	-1.5231	1.00	-1.5231
Total Value:	103.78		Duration:	5.5040

Recall also from Chapter 2 that the price of the leveraged inverse floater can be computed as:[7]

$$\text{Price leveraged inverse floater} P_{LIF}(0,T) = 2 \times P_z(0,T) + P_c(0,T) - 2 \times P_{FR}(0,T) \tag{3.52}$$

To compute the duration of the leveraged inverse floater we need to compute the duration of the fixed-coupon bond. Using the same steps as in Table 3.8 but with coupon rate $c = 25\%$ we find that the duration of the coupon bond in this case is $D_c = 2.5448$. Given this information, we can now compute the duration of the leveraged inverse floater. Table 3.9 contains the calculations. The 3-year leveraged inverse floater has a duration of 5.5040, almost twice its maturity. This security is very sensitive to changes in interest rates, indeed.

3.7.5 What Can We Infer about the Orange County Portfolio?

With these data we can get a sense of the composition of the Orange County portfolio. It appears that the portfolio had about $2.8 billion in "inverse floaters [...], index amortizing notes, and collateralized mortgage obbligations."[8] For simplicity, we assume that $2.8 billion was invested only in leveraged inverse floaters. Assuming the remaining part of the portfolio was invested in safe Treasury securities, what should have the duration of this additional investment been?

Let $x = 2.8/20.5 = 0.1366$ be the fraction of total assets invested in leveraged inverse floaters. Then, we know that

$$\text{Duration of assets} = x \times \text{Duration of leveraged inverse floater} + (1-x) \times \text{Duration of T-bills} \tag{3.53}$$

From $x = 0.1366$, the duration of leveraged inverse floaters ($= 5.5040$) and the duration of assets ($= 2.89$)

$$\text{Duration of T-bills} = \frac{2.89 - 0.1366 \times 5.5040}{1 - 0.1366} = 2.4764 \tag{3.54}$$

That is, the $20.5 billion Orange County portfolio could well have been mainly invested in short-term Treasury bonds (with duration of only 2.4764). Yet, the large leverage and the very high duration of leveraged inverse floaters may still have produced large losses as the interest rate increased.

[7] We are still making the simplifying assumption that we know that $c(t) > 0$ for sure, i.e. that $r_1(t) < 25\%/2$.

[8] See ERisk Case, Orange County (2001), page 2.

3.7.6 Conclusion

In conclusion, this case illustrates the risk embedded in fixed income securities, and, in particular, in leveraged positions. Structured securities, such us leveraged inverse floaters, contain additional risks that the risk manager must be aware of. In particular, this case emphasizes that even if the average maturity of the instruments may be very low, the risk of such securities or portfolio may be very high. In this sense, the interpretation of duration as the weighted average of cash flow payments is strongly misleading. As illustrated in the case, the Orange County portfolio could well have been mainly invested in short-term Treasuries and leveraged inverse floaters. Yet, this portfolio still has a large sensitivity to interest rates, and therefore it is very risky.

3.8 CASE ANALYSIS: THE EX-ANTE RISK IN ORANGE COUNTY'S PORTFOLIO

In hindsight it seems that Orange County's investment strategy paved the way for its own disaster, but any reasonable assessment must be made using *ex ante* information. In particular, was there anything that ex ante could have warned Orange County's Treasurer and its creditors regarding the potential risk that the portfolio was bearing? We can answer this question by using the concepts of Value-at-Risk and expected shortfall introduced in Sections 3.2.8 and 3.2.9, respectively. We compute these risk measures under both the historical distribution and normal distribution approach by making use of all the information available up to January 1994.[9]

1. **Historical Distribution Approach.** We can use the past changes in the level of interest rates dr as a basis to evaluate the potential changes in a portfolio value dP. Panel A of Figure 3.7 shows the historical changes in the average level of interest rates at the monthly frequency. Panel B makes a histogram of these changes, that is, describes the frequency of each possible change. As can be seen, large increases and decreases are not very likely, but they do occur occasionally. We can now multiply each of these changes dr observed in the plot by $-D_P \times P$ to obtain the variation in dP. Figure 3.8 plots the histogram of the changes in the portfolio (i.e. the portfolio profit and loss, or P&L). Given this distribution, we can compute the maximum loss that can occur with 99% probability. We can start from the left-hand side of the distribution, and move right until we count 1% of the observations. That number is the 99% monthly VaR computed using the historical distribution approach. In this case, we find it equal to $715 million. That is, there is only 1% probability that Orange County portfolio could lose more than $715 million in one month. The corresponding expected shortfall, obtained by averaging all of the portfolio losses that are lower than $715 million, turns out to be $990 million. That is, the expected monthly loss of the Orange County portfolio in case of an extreme event is $990 million.

2. **Normal Distribution Approach.** We can also use some assumption about the distribution of interest rates. For instance, if dr is normally distributed, so is dP.

[9]It should be mentioned that by this date Value-at-Risk and expected shortfall were not yet been introduced as risk measures, and therefore Orange County's Treasurer could have not done the following calculations.

From the data in the top panel of Figure 3.7 we can compute the historical mean and standard deviation of dr, and thus obtain the mean and standard deviation of dP. In particular, we find

$$\text{Mean}(dr) = \widehat{\mu}_{dr} = 4.71e - 05; \quad \text{Std}(dr) = \widehat{\sigma}_{dr} = 0.00432;$$

which implies

$$\text{Mean}(dP) = -D_P \times P \times \widehat{\mu}_{dr} = -0.0028; \quad \text{Std}(dP) = D_P \times P \times \widehat{\sigma}_{dr} = 0.2563$$

Figure 3.8 also reports the normal density with mean $\mu_P = -.0028$ and standard deviation $\sigma_{dP} = 0.2563$. In this case, the 99% maximum loss can be computed from the properties of the normal distribution, resulting in VaR $= -(\mu_P - 2.326 \times \sigma_{dP}) =$ $\$598$ million. This number is smaller than the one obtained under the historical distribution approach, because of the fat-tailed distribution of the portfolio P&L, as showin in Figure 3.8: Extreme realizations are more likely under the historical distribution than under the normal distribution. Indeed, from Fact 3.7 the 99% expected shortfall in the case of the normal distribution is only $680 million, which is much smaller than the $990 million expected shortfall obtained under the historical distribution approach.

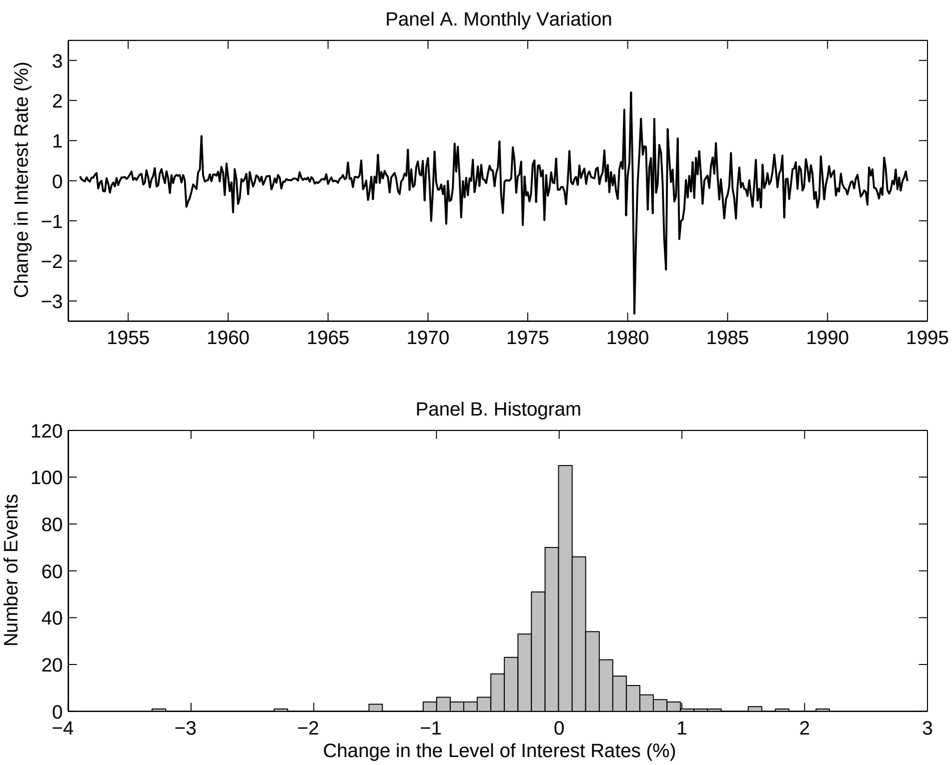
The VaR numbers computed above are relatively small compared to the ex-post $1.6 billion loss. It is important to realize, though, that the VaR so computed is a *monthly* figure, while Orange County losses accrued over a six-month period. How can we compute a 6-month VaR? Using the normal distribution approach and assuming that monthly changes in interest rates are independent and identically distributed – a strong assumption as there is some predictability in yields, as discussed in Chapter 7 – the annualization can be performed by multiplying the mean μ_P by 6 and the monthly standard deviation σ_P by $\sqrt{6}$. In this case we obtain a 99% 6-month VaR equal to $1.48 billion, close to the actual loss suffered by Orange County.

3.8.1 The Importance of the Sampling Period

The VaR calculation is very sensitive to the sample used in the calculation. In fact, the top panel of Figure 3.7 shows that the volatility of the level of interest rates had been relatively low in the decade before 1994. The large estimate of the monthly standard deviation of interest rates $\sigma_{dr} = 0.00432$ is mainly due to the large volatility in the 1970s and beginning of the 1980s. If we restrict the sample to compute the standard deviation of interest rate changes dr to the more recent period, such as five years, we find the much smaller standard deviation $\sigma_{dr} = 0.0028$. The 6-month, 99% VaR in this case is $668 million, a large number, but much smaller than the actual ex-post losses.

What sample period is more relevant? The large shift up in the interest rate was probably unexpected. However, it is a common mistake to confuse a low volatility period as a safe period. History shows time and again that low volatility periods are followed by high volatility periods. Thus, reliance on the recent past would miss the probability that in fact volatility *will* go up, and with it, the risk of large losses. The use of a longer sample that takes into account such facts is therefore more conservative for the risk manager.

Figure 3.7 The Monthly Changes in the Average Level of Interest Rates



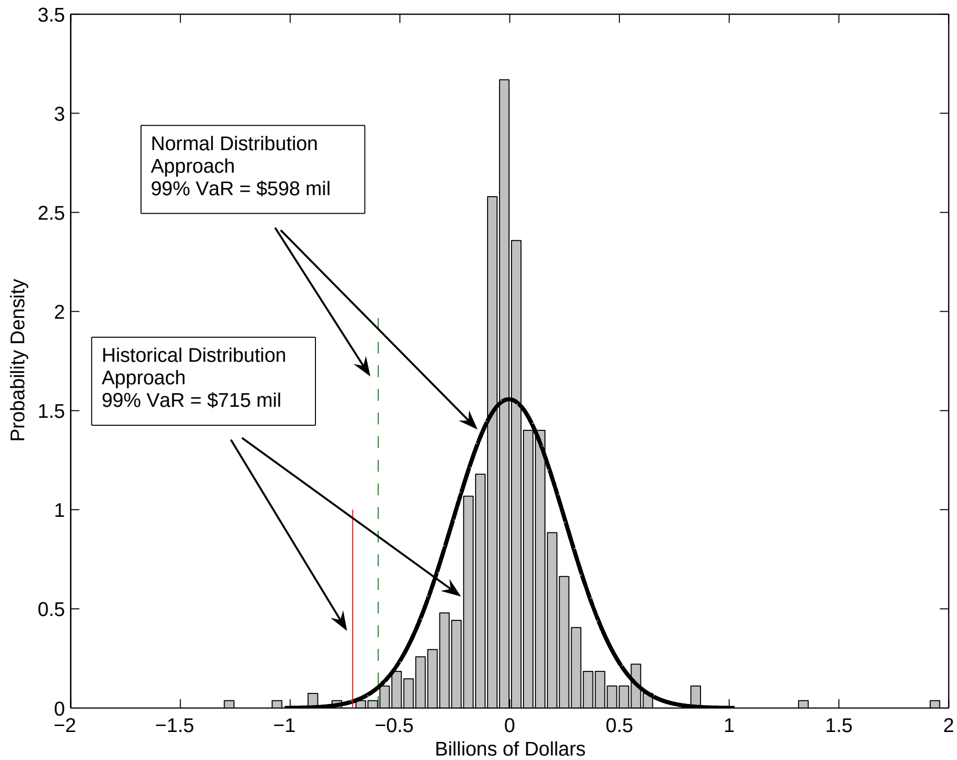
Data Source: CRSP.

3.8.2 Conclusion

The ex-ante measurement of risk is difficult and full of potential pitfalls. In the previous section, we computed several numbers in the attempt to measure the risk embedded in the Orange County portfolio. The numbers we computed vary greatly depending on (a) the type of model (e.g., normal versus historical); (b) the horizon (one month versus six months); (c) the sample used (last five years versus longer sample); (d) the type of risk measure (VaR versus Expected Shortfall).

The natural question is then the following: Which one of these measures is best? Unfortunately, this is hard to tell. While it sounds intuitive that we should always use the most conservative measure of risk, i.e. the one implying the largest possible losses, there are good reasons also to not be too conservative, as a portfolio manager who is overly conservative may miss important profit opportunities. The main goal of this case is to show that the measurement itself of risk is difficult, and so the portfolio manager should always be suspicious of any risk measure, and always ask *how* such a measure was calculated. The major risk for a portfolio manager is in fact to rely too much on these measures of risk,

Figure 3.8 The Distribution of the Monthly P&L of the Orange County Portfolio



forgetting that they are fragile, in the sense that they greatly depend on the way they are computed.

3.9 APPENDIX: EXPECTED SHORTFALL UNDER THE NORMAL DISTRIBUTION

In this appendix we derive the formula in Equation 3.34 for the expected shortfall under the normal distribution case. Let dP denote the P&L of the portfolio, so that the loss is $L_T = -dP$. Clearly, $L_T > VaR$ when $dP < -VaR$. Let z be the "quantile" corresponding to the VaR. For instance, $z = 1.645$ for the 95% VaR, and recall from Equation 3.30, $dP \sim N(\mu_P, \sigma_P)$. We then have

$$\begin{aligned} E\left[L_T | L_T > VaR\right] &= -E\left[dP | dP < -VaR\right] = -E\left[dP | dP < (\mu_P - \sigma_P \times z)\right] \\ &= -\left\{\mu_P + \sigma_P \times E\left[\left(\frac{dP - \mu_P}{\sigma_P}\right) | \left(\frac{dP - \mu_P}{\sigma_P}\right) < -z\right]\right\} \end{aligned} \quad (3.55)$$

The quantity $X = \left(\frac{dP-\mu_P}{\sigma_P}\right)$ is a standardized normal distribution. Since

$$E[X|X < -z] = \frac{\int_{\infty}^{-z} x f(x) dx}{\int_{\infty}^{-z} f(x) dx} = \frac{-f(-z)}{\mathcal{N}(-z)}$$

the formula in Equation 3.34 follows from substituting this latter expression into Equation 3.55.